GENERIC VANISHING, PLURI-CANONICAL MAPS AND VOLUME OF ISOLATED SINGULARITY

by

Yuchen Zhang

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

The University of Utah

May 2014

The University of Utah Graduate School

STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Yuchen Zhang**

has been approved by the following supervisory committee members:

Christopher Hacon	, Chair	**03/06/2014** Date Approved
Tommaso de Fernex	, Member	**03/06/2014** Date Approved
Aaron Bertram	, Member	**03/06/2014** Date Approved
Peter Trapa	, Member	**03/06/2014** Date Approved
Ilya Zharov	, Member	Date Approved

and by **Peter Trapa**, Chair/Dean of

the Department/College/School of **Mathematics**

and by David B. Kieda, Dean of The Graduate School.

ABSTRACT

In this thesis, we consider two different problems in birational geometry considered previously in the author's papers.

The first problem concerns pluri-canonical maps in positive characteristic. We prove that for a smooth variety X of general type over an algebraically closed field k with positive characteristic, if X has maximal Albanese dimension and the Albanese map is separable, then $|4K_X|$ induces a birational map.

The second problem is on the volume of isolated singularities over $\mathbb{C}$. We give an equivalent definition of the local volume of an isolated singularity $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)$ defined by Boucksom, de Fernex and Favre in the $\mathbb{Q}$-Gorenstein case and we generalize it to the non-$\mathbb{Q}$-Gorenstein case. We prove that there is a positive lower bound depending only on the dimension for the non-zero local volume of an isolated singularity if X is Gorenstein. We also give a non-$\mathbb{Q}$-Gorenstein example with $\mathrm{Vol}_{\mathrm{BdFF}}(X,0)=0$, which does not allow a boundary Δ such that the pair (X,Δ) is log canonical.

CONTENTS

NOTATION AND SYMBOLS

We will use the standard notations in [13, 17, 18].

$\hat{A}$	Dual abelian variety of A
P	Normalized Poincare line bundle
$D(X)$	Derived category of coherent sheaves on X
$R\hat{S}$, RS	Fourier-Mukai transforms
ω_X	Canonical line bundle on X
F	Absolute Frobenius morphism
Tr	Trace map $F_*\omega_X \to \omega_X$
$\tau(\mathfrak{a}^\lambda)$	Test ideal of $\mathfrak{a}$ with index λ
$\tau(\mathfrak{a}_\bullet^\lambda)$	Asymptotic test ideal of $\{\mathfrak{a}_\bullet\}$ with index λ
$\tau(\lambda \cdot \|D\|)$	Asymptotic test ideal of D with index λ
$f^\natural(D)$	Natural pullback
D_Y	Strict transform of D on Y
E_f	Reduced exceptional divisor of f
$K_{m,Y/X}$	m-th limiting relative canonical $\mathbb{Q}$-divisor
$K_{Y/X}$	Relative canonical $\mathbb{R}$-divisor
$K^\Delta_{Y/X}$	Relative canonical $\mathbb{Q}$-divisor of (X, Δ)
$A_{m,Y/X}$	m-th limiting log discrepancy $\mathbb{Q}$-divisor
$A_{Y/X}$	Log discrepancy $\mathbb{R}$-divisor
$A^\Delta_{Y/X}$	Log discrepancy $\mathbb{Q}$-divisor of (X, Δ)
$a(F; X, \Delta)$	Log discrepancy of F with respect to (X, Δ)
$\mathrm{Div}(\mathcal{X})$	Weil b-divisors over X
$\mathrm{CDiv}(\mathcal{X})$	Cartier b-divisors over X
$\overline{D}$	Cartier b-divisor determined by pulling back D

$\mathrm{Env}_{\mathcal{X}}(D)$	Nef envelope of an $\mathbb{R}$-divisor D
$\mathrm{Env}_{\mathcal{X}}(W)$	Nef envelope of an $\mathbb{R}$-Weil b-divisor W
$\mathrm{Vol}_{BdFF}(X,0)$	Non-log-canonical volume defined in [3]
$\mathrm{Vol}_m(X)$	m-th limiting volume
$\mathrm{Vol}^+(X)$	Augmented volume

ACKNOWLEDGEMENTS

First of all, I am deeply indebted to my Ph.D. advisor, C. Hacon, who taught me a lot of mathematics, suggested several interesting problems and gave helpful suggestions to their solutions.

Secondly, I would like to thank the professors in Utah, A. Bertram and Y.P. Lee, and especially T. de Fernex and C. Xu, who gave a lot of help in my work on the volume of isolated singularities.

I also thank for many useful mathematical conversations with M. Fulger, M. Mustaţă, Z. Patakfalvi, K. Schwede, S. Takagi and K. Tucker. I have to thank all my follow Ph.D. students, especially O. Das and my coauthor, A. Watson, for the friendly environment and ultimate atmosphere for mathematical discussions.

Finally, I would like to thank my parents who led me to mathematics and supported me always. In particular, I am grateful to my girlfriend, Mengyao, who always encouraged me when I was frustrated or was lazy.

Thank you all!

CHAPTER 1

INTRODUCTION

In this thesis, we will consider the following two problems in birational geometry:

- It is known that for a smooth variety X of general type, the linear series $|mK_X|$ induce a birational map when m is sufficiently large. Is there a bound for such m?
- A volume is defined in [3] to study isolated singularities, especially when K_X is not $\mathbb{Q}$-Cartier. It is known that if (X, Δ) is log canonical for some boundary Δ, then the volume is 0. Is the converse statement valid?

Let X be a smooth variety of general type, i.e., the canonical divisor K_X is big. We may consider the map associated to the linear series $|mK_X|$ for any positive integer m.

$$\phi_{|mK_X|} : X \dashrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(mK_X)).$$

Since for any ample divisor H we can find a positive integer m and write $mK_X \sim H + E$ for some effective divisor E, the natural map $\phi_{|mK_X|}$ above is birational for any m sufficiently large. It is natural to ask if a bound can be found for such m.

When X lies over the complex number field $\mathbb{C}$, it is shown by Hacon and McKernan [11], and independently by Takayama [25] and by Tsuji [26, 27], that there is a bound m only depending on the dimension of X such that for any $n \geqslant m$, the map $\phi_{|nK_X|}$ is birational. More precisely, the optimal bound for curves is 3 which is an easy consequence of the Riemann-Roch Theorem. The optimal bound for surfaces is 5 shown by Bombieri [2]. The optimal bound for 3-folds is still unknown.

The study of pluri-canonical maps for irregular varieties was started by Chen and Hacon. Abelian varieties are one of the most studied objects in algebraic geometry. For irregular varieties X, we may consider the Albanese morphism

$$\mathrm{alb}_X : X \to \mathrm{Alb}_X.$$

In characteristic 0, the Albanese variety is an abelian variety given by

$$\mathrm{Alb}_X = \frac{H^0(X, \Omega_X^1)^\vee}{H^1(X, \mathbb{Z})}.$$

In positive characteristic, the Albanese morphism is defined by the universal property that any other morphism from X to an abelian variety must factor through alb_X. The dimension of the image of alb_X is called the Albanese dimension. We say that a variety X has maximal Albanese dimension if the Albanese dimension is the same as the dimension of X. For complex projective varieties having maximal Albanese dimension, it is shown in [5] that $|3K_X|$ is birational for varieties with positive Euler characteristic and $|6K_X|$ is birational for any varieties. The bound is refined to 5 by Jiang [14]. Finally, the optimal bound 3 is obtained by Jiang, Lahoz and Tirabassi [15]. Furthermore, it is shown in [5] that if the Albanese dimension is $\dim X - 1$, then $|6K_X|$ is birational, and if the Albanese dimension is $\dim X - 2$, then $|7K_X|$ is birational.

All the results listed above are valid only in characteristic 0. Surprisingly, little is known in positive characteristic. In this thesis, we will prove the following theorem which first appears in [31] by the author:

Theorem 1 *(See Theorem 17) Let X be a smooth projective variety of general type over an algebraically closed field k of characteristic $p > 0$. If X has maximal Albanese dimension and the Albanese map is separable, then $|4K_X|$ induces a birational map.*

The main tool used in the study of abelian varieties is the Fourier-Mukai transform introduced by Mukai [19]. Fortunately, this theory still applies in positive characteristic. However, in order to produce sections in the linear series $|mK_X|$, the references mentioned above used multiplier ideals and Kawamata-Viehweg vanishing in an essential way. In positive characteristic, Kawamata-Viehweg vanishing is known to fail. Inspired by [10], [20] and [23], we replace Kawamata-Viehweg vanishing by the Frobenius map and Serre vanishing. Combining this with the Fourier-Mukai transform, we obtain that $|4K_X|$ is birational. It seems that new ideas are required to investigate the third pluri-canonical map.

The second topic in this thesis is the non-log-canonical volume $\mathrm{Vol}_{\mathrm{BdFF}}$ defined by Boucksom, de Fernex and Favre [3] for varieties with isolated singularities over $\mathbb{C}$. The story traces back to the local volume for normal surfaces defined by Wahl [29]. Let X be a normal surface and $f : Y \to X$ be the minimal resolution. We consider the relative Zariski decomposition over X,

$$A_{Y/X} = K_Y + E - f^* K_X = P + N,$$

where E is the reduced exceptional divisor, P is f-nef and N is an effective f-exceptional divisor. The local volume is defined as

$$\mathrm{Vol}(X) = -P^2.$$

It can be proved that the volume is 0 if and only if there is a boundary Δ such that (X, Δ) is log canonical. It is also shown by Ganter [9] that the least possible positive volume is $1/42$. In dimension 2, Wahl's local volume for normal surfaces plays an essential role in the classification of projective surfaces admitting noninvertible endomorphisms, which is now essentially complete [7, 21].

In [3], Boucksom, de Fernex and Favre generalize Wahl's local volume to higher dimensions and use it to study the noninvertible endomorphism of isolated singularities. Specifically, they show that if $(X, 0)$ allows an endomorphism preserving the singularity of degree greater than 1, then $\mathrm{Vol}_{\mathrm{BdFF}} = 0$. Moreover, if X is $\mathbb{Q}$-Gorenstein, then $\mathrm{Vol}_{\mathrm{BdFF}} = 0$ if and only if X has log canonical singularities. For a better understanding of $\mathrm{Vol}_{\mathrm{BdFF}}$, they propose two questions:

Problem A Does there exist a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume?

Problem B Is it true that $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$ implies the existence of an effective $\mathbb{Q}$-boundary Δ such that the pair (X, Δ) is log-canonical (the converse being easily shown)?

These two problems are studied by the author in [30]. For the first problem, we will give an alternate definition of $\mathrm{Vol}_{\mathrm{BdFF}}$ in the $\mathbb{Q}$-Gorenstein case. We will prove that the positive part in the relative Zariski decomposition defined in [3] is a $\mathbb{Q}$-Cartier b-divisor given by the pullback of the log discrepancy divisor on the log canonical modification. As a result, the volume can be calculated on a fixed model. The existence of log canonical modification is proved by Odaka and Xu [22]. Combining with the descending chain condition (DCC) for the volumes proved by Hacon, McKernan and Xu [12], we will give a positive answer to Problem A.

Theorem 2 *(See Theorem 29) There exists a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume.*

For varieties which are not $\mathbb{Q}$-Gorenstein, the traditional approach is to find a $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and to study the pair (X, Δ). In [6], de Fernex and

Hacon define pullback of divisors by proper birational morphisms which are possibly not $\mathbb{Q}$-Cartier. They suggest that we can study the singularities via the relative canonical divisor $K_{Y/X} = K_Y - f^*K_X$ directly instead of working with pairs. For example, we can define X to be canonical if $K_{Y/X} \geqslant 0$ for any log resolution Y. It is natural to ask if the new definition is compatible to the traditional one with pairs. For example:

Question: Are the following conditions equivalent to each other?

1. For any exceptional divisor F on any log resolution Y of X,
$$\operatorname{ord}_F(K_{Y/X}) \geqslant -1.$$
2. There is a boundary Δ such that (X, Δ) is log canonical.

Unfortunately, this is wrong even in dimension 3. We will give a counterexample in Section 4.2.1. This is also a counterexample of Problem B proposed in [3].

Theorem 3 *(See Theorem 38) There exists a polarized smooth variety (V, H) such that the affine cone $X = C(V, H)$ has volume $\mathrm{Vol}_{BdFF}(X, 0) = 0$, but there is no boundary Δ such that (X, Δ) is log canonical.*

We also give an alternate definition of $\mathrm{Vol}_{\mathrm{BdFF}}$ in the non-$\mathbb{Q}$-Gorenstein case using the log canonical modification. We will give the definition of the augment volume $\mathrm{Vol}^+(X)$ by finding a suitable boundary. It can be proved that

$$\mathrm{Vol}^+(X) \geqslant \mathrm{Vol}_{\mathrm{BdFF}}(X).$$

However, we do not know if these two volumes are the same or not in general.

It should be noticed that, Fulger [8] defined another volume Vol_F for isolated singularities using local cohomology. It is shown that $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) \geqslant \mathrm{Vol}_{\mathrm{F}}(X, 0)$ with equality if X is $\mathbb{Q}$-Gorenstein. In [3, Example 5.4], an example is given where $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) > \mathrm{Vol}_{\mathrm{F}}(X, 0)$.

In the spirit of [6], one should approach the non-$\mathbb{Q}$-Gorenstein case without the boundary. It is conjectured that for a normal variety X (possibly not $\mathbb{Q}$-Gorenstein) which has only isolated singularities, there is a log canonical modification $f : Y \to X$ in the sense that $K_Y + E_f$ is f-ample and (Y, E_f) is log canonical. In [4, Proposition 2.4], it is proved that if such modifications exist, then $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$ if and only if f is an isomorphism in codimension 1.

This thesis is organized as follows. In Chapter 2, we give the related definitions such as Fourier-Mukai transforms, test ideals and nef envelopes. We list basic properties and lemmas. For those lemmas that we will use in an essential way, we also include the proofs. In Chapter 3, we focus on the pluri-canonical maps in positive characteristic. In the last chapter, we study the local volumes.

CHAPTER 2

BACKGROUND MATERIAL

In this thesis, we will use A to denote an abelian variety and X to denote a normal variety over an algebraically closed field k.

2.1 Fourier-Mukai Transforms

One of the main technical tools applied in this thesis is the Fourier-Mukai transform first introduced by Mukai in [19]. Let $\hat{A}$ be the dual abelian variety of A. Let P be the normalized Poincare line bundle on $A \times \hat{A}$ such that

1. for any $a \in A$, $P|_{\{a\}\times\hat{A}} = P_a$ is a topologically trivial line bundle on $\hat{A}$,

2. for any $\hat{a} \in \hat{A}$, $P|_{A\times\{\hat{a}\}} = P_{\hat{a}}$ is a topologically trivial line bundle on A,

3. $P_0 = \mathcal{O}_{\hat{A}}$ and $P_{\hat{0}} = \mathcal{O}_A$.

Let p_A and $p_{\hat{A}}$ be the projections from $A \times \hat{A}$ to A and $\hat{A}$, respectively. Let $\hat{S}$ be the functor between the categories coherent sheaves on A and $\hat{A}$ defined as:

$$\hat{S}(\mathcal{F}) = p_{\hat{A},*}(p_A^*\mathcal{F} \otimes P).$$

The Fourier-Mukai transform $R\hat{S} : D(A) \to D(\hat{A})$ is the right derived functor of $\hat{S}$. Similarly, we define $RS : D(\hat{A}) \to D(A)$ as the right derived functor of $S(\mathcal{G}) = p_{A,*}(p_{\hat{A}}^*\mathcal{G} \otimes P)$. These functors are equivalences of triangulated categories $D(A)$ and $D(\hat{A})$. Indeed, Mukai proved the following theorem [19, Theorem 2.2]:

Theorem 4 *The following properties hold on $D(A)$ and $D(\hat{A})$.*

$$RS \circ R\hat{S} = (-1_A)^*[-g] \qquad R\hat{S} \circ RS = (-1_{\hat{A}})^*[-g],$$

where -1_A is the inverse on A and $[-g]$ denotes the shift by g places to the right.

Example 5 *It is easy to calculate that*

$$R\hat{S}(k(0)) = Rp_{\hat{A},*}(p_A^* k(0) \otimes P) = Rp_{\hat{A},*}(P_{\{0\}\times\hat{A}}) = \mathcal{O}_{\hat{A}}$$

and

$$RS(\mathcal{O}_{\hat{A}}) = Rp_{A,*}(p_{\hat{A}}^* \mathcal{O}_{\hat{A}} \otimes P) = Rp_{A,*}(P) = k(0)[-g].$$

We denote with $R^i\hat{S}(\mathcal{F})$ (respectively $R^iS(\mathcal{F})$) the i-th cohomology group of the complex $R\hat{S}(\mathcal{F})$ (respectively $RS(\mathcal{F})$). Then we can define the following [19, Definition 2.3]:

Definition 6 *We say that the Weak Index Theorem with index i (WIT_i) holds for an object $\mathcal{F}$ in $D(A)$ if $R^j\hat{S}(\mathcal{F}) = 0$ for all $j \neq i$. We denote the coherent sheaf $R^i\hat{S}(\mathcal{F})$ by $\hat{\mathcal{F}}$ and call it the Fourier-Mukai transform of $\mathcal{F}$.*

We say that the Index Theorem with index i (IT_i) holds for a coherent sheaf $\mathcal{F}$ on A if for any $\hat{a} \in \hat{A}$ and any $j \neq i$, we have $H^i(A, \mathcal{F} \otimes P_{\hat{a}}) = 0$.

It can be proved using base change that a coherent sheaf $\mathcal{F}$ satisfies IT_i if and only if it satisfies WIT_i and $\hat{\mathcal{F}}$ is a vector bundle.

A very nice application of the Fourier-Mukai transform is due to Chen and Hacon first appearing in [5]. This proposition plays an essential role in the proof of Theorem 17.

Proposition 7 *Let $\mathcal{F}$ be a nonzero coherent sheaf on A satisfying IT_0. If $\mathcal{F} \to k(a)$ is a surjective morphism for some $a \in A$, then the induced map*

$$H^0(A, \mathcal{F} \otimes P_{\hat{a}}) \to H^0(A, k(a) \otimes P_{\hat{a}}) \cong k(a)$$

is surjective for general $\hat{a} \in \hat{A}$.

Proof. We quote the proof from [10, Proposition 2.1] which can also be found in [5]. Since $\mathcal{F}$ satisfies IT_0, we have that $RS(\hat{F}) = (-1_A)^*\mathcal{F}[-g] \neq 0$ by Theorem 4, thus $\hat{F} \neq 0$. Since $P_a = R^0\hat{S}(k(a)) = R\hat{S}(k(a))$, the homomorphism $\phi : \hat{\mathcal{F}} \to P_a$ is nonzero. As P_a is a line bundle, it follows that ϕ is generically surjective. The proposition follows from cohomology and base change. ∎

2.2 Asymptotic Test Ideals

Suppose that X is a smooth n-dimensional variety over an algebraically closed field k of characteristic $p > 0$. Let ω_X denote the canonical line bundle on X. We denote $F : X \to X$ the absolute Frobenius morphism, that is given by the identity on the topological space, and by taking the p-th power on regular functions. Let $\mathrm{Tr} : F_*\omega_X \to \omega_X$ be the trace map. In local coordinates, the trace map is characterized by

$$\mathrm{Tr}(x_1^{i_1} \cdots x_n^{i_n} dx_1 \wedge \cdots \wedge dx_n) = x_1^{\frac{i_1+1}{p}-1} \cdots x_n^{\frac{i_n+1}{p}-1} dx_1 \wedge \cdots \wedge dx_n,$$

where $x_k^{\frac{i_k+1}{p}-1} = 0$ if p does not divide $i_k + 1$. Let $\mathrm{Tr}^e : F_*^e\omega_X \to \omega_X$ be the e-th iteration of the trace map.

We follow the definitions given in [20]. For other equivalent definitions, see [1] and [23]. Given a nonzero ideal $\mathfrak{a}$ in $\mathcal{O}_X$, the image $\mathrm{Tr}^e(F_*^e(\mathfrak{a} \cdot \omega_X))$ can be written as $\mathfrak{a}^{[1/p^e]} \cdot \omega_X$ for some ideal $\mathfrak{a}^{[1/p^e]}$ in $\mathcal{O}_X$. Given a positive real number λ, one can show that

$$\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]} \subseteq \left(\mathfrak{a}^{\lceil \lambda p^{e+1} \rceil}\right)^{[1/p^{e+1}]}$$

for every $e \geqslant 1$ where $\lceil t \rceil$ means the smallest integer $\geqslant t$. Since X is Noetherian, there is an ideal $\tau(\mathfrak{a}^\lambda)$, called the test ideal of $\mathfrak{a}$ of exponent λ, that is equal to $\left(\mathfrak{a}^{\lceil \lambda p^e \rceil}\right)^{[1/p^e]}$ for all e large enough.

Test ideals have many similar properties to multiplier ideals. If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^\lambda) \subseteq \tau(\mathfrak{b}^\lambda)$ for all $\lambda \geqslant 0$. If m is a positive integer, then $\tau(\mathfrak{a}^{m\lambda}) = \tau((\mathfrak{a}^m)^\lambda)$.

One can also define an asymptotic version of test ideals similar to asymptotic multiplier ideals. Suppose that $\{\mathfrak{a}_\bullet\}$ is a graded sequence of ideals on X ($\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$) and λ is a positive real number. If m and l are two positive integers such that $\mathfrak{a}_m$ is nonzero, then

$$\tau(\mathfrak{a}_m^{\lambda/m}) = \tau((\mathfrak{a}_m^l)^{\lambda/ml}) \subseteq \tau(\mathfrak{a}_{ml}^{\lambda/ml}).$$

By the Noetherian property, there is a unique ideal $\tau(\mathfrak{a}_\bullet^\lambda)$, called the asymptotic test ideal of $\{\mathfrak{a}_\bullet\}$ of exponent λ, such that $\tau(\mathfrak{a}_\bullet^\lambda) = \tau(\mathfrak{a}_m^{\lambda/m})$ for all m large enough and sufficiently divisible.

For linear series, let D be a Cartier divisor on X such that $h^0(X, \mathcal{O}_X(mD)) \neq 0$ for some positive integer m. We then define $\tau(\lambda \cdot \|D\|) = \tau(\mathfrak{a}_\bullet^\lambda)$ where $\mathfrak{a}_m$ is the base ideal of the linear series $|mD|$. Then by definition, $\tau(\lambda/r \cdot \|rD\|) = \tau(\lambda \cdot \|D\|)$ for every positive integer r. If D is a $\mathbb{Q}$-divisor such that $h^0(X, \mathcal{O}_X(mD)) \neq 0$ for some positive integer m satisfying that mD is Cartier, then we put $\tau(\lambda \cdot \|D\|) = \tau(\lambda/r \cdot \|rD\|)$ for some $r > 0$ such that rD is Cartier.

2.3 Singularity For Pairs

Let X be a normal variety. A $\mathbb{Q}$-Weil divisor Δ on X is called a boundary if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\lfloor \Delta \rfloor = 0$. In this case, we say that (X, Δ) is a pair. A pair (X, Δ) has simple normal crossings if

1. X is smooth,

2. each irreducible component of $\mathrm{Supp}(\Delta)$ is smooth, and

3. locally analytically, $\mathrm{Supp}(\Delta) \subset X$ is isomorphic to the intersection of coordinate hyperplanes in affine space $\{x_1 x_2 \cdots x_r = 0\} \subset \mathbb{A}^n$.

A log resolution of a pair (X, Δ) is a proper birational morphism $f : Y \to X$ such that $\mathrm{Ex}(f)$ is a divisor and $(Y, \Delta_Y + \mathrm{Ex}(f))$ has simple normal crossings, where Δ_Y is the strict transform of Δ on Y and $\mathrm{Ex}(f)$ is the exceptional set of f.

We will write

$$A^{\Delta}_{Y/X} = K_Y + \Delta_Y + E_f - f^*(K_X + \Delta),$$

where $f_* K_Y = K_X$ and E_f is the reduced exceptional divisor. Let F be any prime exceptional divisor on Y. The log discrepancy of F with respect to (X, Δ) is given by

$$a(F; X, \Delta) = \mathrm{mult}_F(A^{\Delta}_{Y/X}).$$

We say a divisor F lies over X if there is a proper birational morphism $f : Y \to X$ and F is a divisor on Y. A pair (X, Δ) is called log canonical (resp. Kawamata log terminal or klt) if for any prime exceptional divisor F over X, we have that $a(F; X, \Delta) \geqslant 0$ (resp. $a(F; X, \Delta) > 0$). It suffices to check the exceptional divisors on any given log resolution of (X, Δ).

2.3.1 Log canonical modification

Suppose (X, Δ) is a pair such that X is a normal variety, Δ is an effective $\mathbb{Q}$-divisor and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A birational projective morphism $f : Y \to X$ is called a log canonical modification of (X, Δ) if

1. $(Y, \Delta_Y + E_f)$ is log canonical,

2. $K_Y + \Delta_Y + E_f$ is f-ample,

where Δ_Y is the strict transform of Δ and E_f is the reduced exceptional divisor of f. It is shown in [22] that the log canonical modification exists uniquely up to isomorphism for any log pair (X, Δ). Indeed, let $f' : Y' \to X$ be a log resolution of the pair (X, Δ). We run the relative minimal model problem for the pair $(Y', \Delta_{Y'} + E_{f'})$ over X and get $f_{min} : Y_{min} \to X$ such that $(Y_{min}, \Delta_{Y_{min}} + E_{f_{min}})$ is klt and that $K_{Y_{min}} + \Delta_{Y_{min}} + E_{f_{min}}$ is f_{min}-nef. In [22], Odaka and Xu showed that the pair $(Y', \Delta_{Y'} + E_{f'})$ has a good minimal model. Thus, we may assume $K_{Y_{min}} + \Delta_{Y_{min}} + E_{f_{min}}$ is f_{min}-semi-ample. In particular, the canonical ring

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} f_{min,*}\mathcal{O}_{Y_{min}}(m(K_{Y_{min}} + \Delta_{Y_{min}} + E_{f_{min}}))$$

is finitely generated. Therefore, the log canonical modification

$$Y \cong \mathbf{Proj}_X \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} f_{min,*}\mathcal{O}_{Y_{min}}(m(K_{Y_{min}} + \Delta_{Y_{min}} + E_{f_{min}}))$$

exists. Clearly, for $f' : Y' \to X$, any log resolution of the pair (X, Δ), we have the log canonical modification

$$Y \cong \mathbf{Proj}_X \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} f'_*\mathcal{O}_{Y'}(m(K_{Y'} + \Delta_{Y'} + E_{f'})).$$

The uniqueness follows.

We will need the following lemma in the future.

Lemma 8 *Let (X, Δ) be a pair as above which is not log canonical. Let $f : Y \to X$ be the log canonical model. Write $f^*(K_X + \Delta) \sim_{\mathbb{Q}} K_Y + \Delta_Y + B$, and $B = \sum b_i B_i$ as the sum of distinct prime divisors such that $f_*(B) = \Delta$. We let $B^{>1}$ be the nonzero divisor $\sum_{b_i > 1} b_i B_i$, then $\mathrm{Supp}(B^{>1}) = \mathrm{Ex}(f)$. In particular, $\mathrm{Ex}(f) \subset Y$ is of pure codimension 1.*

Proof. See [22, Lemma 2.4]. ∎

2.4 Non-$\mathbb{Q}$-Gorenstein Varieties

The traditional theory of singularities for non-$\mathbb{Q}$-Gorenstein varieties is based on pairs as in the previous section. In [6], de Fernex and Hacon define the pullback of Weil divisors via proper birational morphisms which generalized Mumford's numerical pullback, and hence make it possible to study the singularities without the boundary.

We recall the following definitions from [6].

2.4.1 Valuations of $\mathbb{Q}$-divisors

Let X be a normal variety over $\mathbb{C}$. A divisorial valuation v on X is a discrete valuation of the function field of X of the form $v = q\,\mathrm{val}_F$ where q is a positive integer and F is a prime divisor over X. Let $\mathcal{J} \subset \mathcal{K}$ be a finitely generated sub-$\mathcal{O}_X$-module of the constant sheaf of rational functions $\mathcal{K} = \mathcal{K}_X$ on X. For short, we will refer to $\mathcal{J}$ as a fractional ideal sheaf on X.

The valuation $v(\mathcal{J})$ of a nonzero fractional ideal sheaf $\mathcal{J} \subset \mathcal{K}$ along v is given by

$$v(\mathcal{J}) = \min\{v(\phi) | \phi \in \mathcal{J}(U), U \cap c_X(v) \neq \emptyset\}.$$

The valuation $v(I)$ of a formal linear combination $I = \sum a_k \cdot \mathcal{J}_k$ of fractional ideal sheaves $\mathcal{J}_k \subset \mathcal{K}$ along v is defined by $v(I) = \sum a_k \cdot v(\mathcal{J}_k)$, where a_k are real numbers.

The $\natural$-valuation (or natural valuation) along v of a $\mathbb{R}$-Weil divisor D on X is $v^\natural(D) = v(\mathcal{O}_X(-D)) = v(\mathcal{O}_X(\lfloor -D \rfloor))$. If C is Cartier, then we have that $v^\natural(C) = v(C)$ and $v^\natural(C + D) = v(C) + v^\natural(D)$. Note also that, as $\mathcal{O}_X(D) \cdot \mathcal{O}_X(-D) \subseteq \mathcal{O}_X$, we have that $v^\natural(D) + v^\natural(-D) \geqslant 0$.

To any nontrivial fractional ideal sheaf $\mathcal{J}$ on X, we associate the divisor $\mathrm{div}(\mathcal{J}) = \sum \mathrm{val}_E^\natural(\mathcal{J}) \cdot E$, where the sum is taken over all prime divisors E on X. Consider now a proper birational morphism $f : Y \to X$ from a normal variety Y. For any divisor D on X, the $\natural$-pullback (or natural pullback) of D to Y is given by $f^\natural D = \mathrm{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)$. In the other words, $f^\natural D = \sum \mathrm{val}_E^\natural(D) \cdot E$, where the sum is taken over all prime divisors E on Y. In particular, $\mathcal{O}_Y(-f^\natural D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}$.

We have the following proposition:

Proposition 9 *For every divisor D on X and every positive integer m,*

$$m \cdot v^\natural(D) \geqslant v^\natural(mD).$$

Proof. We quote the following proof from [6, Lemma 2.8]. If $f_1, \ldots, f_m \in \mathcal{O}_X(-D)(U)$ for some open set $U \subseteq X$, then $\mathrm{div}(f_i) \geqslant D$ on U for each i. Thus, $\mathrm{div}(\prod f_i) \geqslant mD$ on U, which means that $\prod f_i \in \mathcal{O}_X(-mD)(U)$. Hence, $\mathcal{O}_X(-D)^m \subseteq \mathcal{O}_X(-mD)$ and the proposition follows. ■

It follows from the above proposition that

$$\inf_{k \geqslant 1} \frac{v^\natural(kD)}{k} = \liminf_{k \to \infty} \frac{v^\natural(kD)}{k} = \lim_{k \to \infty} \frac{v^\natural(k!D)}{k!} \in \mathbb{R}.$$

Let D be a $\mathbb{R}$-divisor on X. We define the valuation along v of D by

$$v(D) = \lim_{k \to \infty} \frac{v^\natural(k!D)}{k!} \in \mathbb{R}.$$

Remark 10 *It is not hard to see that actually*

$$v(D) = \lim_{k \to \infty} \frac{v^\natural(kD)}{k} \in \mathbb{R}.$$

See [3, Proposition 2.1].

Remark 11 *Even if D is a $\mathbb{Q}$-divisor on X, the valuation $v(D)$ may not be a rational number. See [28, Section 3].*

If $f : Y \to X$ is a birational morphism from a normal variety Y, then the pullback of D to Y is defined by

$$f^*D = \sum \mathrm{val}_E(D) \cdot E,$$

where the sum is taken over all prime divisors E on Y. Notice that if D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and m is a positive integer such that mD is Cartier, then

$$v(D) = \frac{v(mD)}{m} \quad \text{and} \quad f^*D = \frac{f^*(mD)}{m},$$

which coincide with the usual valuation and pullback of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. If C is $\mathbb{Q}$-Cartier, then $v(C+D) = v(C) + v(D)$ and $f^*(C+D) = f^*C + f^*D$.

In [6, Lemma 2.7], de Fernex and Hacon proved the following lemma regarding to the composition of pullbacks:

Lemma 12 *Let $f : Y \to X$ and $g : V \to Y$ be two birational morphisms of normal varieties. Then, for every divisor D on X, the divisor $(f \circ g)^\natural D - g^\natural(f^\natural D)$ is effective and g-exceptional. Moreover, if $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is an invertible sheaf, then $(f \circ g)^\natural D = g^\natural(f^\natural D)$. The similar statement applies to f^* and g^*.*

Proof. We quote the following proof from [6, Lemma 2.7 and Remark 2.13]. Notice that for every Cartier divisor C,

$$(f \circ g)^\natural(C+D) - g^\natural(f^\natural(C+D)) = (f \circ g)^\natural D - g^\natural(f^\natural D).$$

By restricting to an open subset and replacing D with $C + D$ for some Cartier divisor $C \geqslant -D$, we may assume that D is effective. Then it suffices to observe that

$$\mathcal{O}_X(-D) \cdot \mathcal{O}_Y \subseteq \mathcal{O}_U(-f^\natural D),$$

with equality holding when $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is an invertible sheaf. ∎

2.4.2 Relative canonical divisors

We recall that a canonical divisor K_X on a normal variety X is, by definition, the (componentwise) closure of any canonical divisor of the regular locus of X. We also recall that X is said to be $\mathbb{Q}$-Gorenstein if some (equivalently, every) canonical divisor K_X is $\mathbb{Q}$-Cartier. For a proper birational morphism $f : Y \to X$ of normal varieties, we fix a canonical divisor K_Y on Y such that $f_*K_Y = K_X$. For any divisor D on X, we will write D_Y for the strict transform $f_*^{-1}D$ of D on Y.

For every $m \geqslant 1$, the m-th limiting relative canonical $\mathbb{Q}$-divisor $K_{m,Y/X}$ of Y over X is

$$K_{m,Y/X} = K_Y - \frac{1}{m} f^\natural(mK_X).$$

The relative canonical $\mathbb{R}$-divisor $K_{Y/X}$ of Y over X is

$$K_{Y/X} = K_Y - f^*K_X.$$

Clearly, $K_{Y/X}$ is the limsup of the $\mathbb{Q}$-divisors $K_{m,Y/X}$. A $\mathbb{Q}$-divisor Δ on X is said to be a boundary, if $\lfloor \Delta \rfloor = 0$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The log relative canonical $\mathbb{Q}$-divisor of (Y, Δ_Y) over (X, Δ) is given by

$$K^{\Delta}_{Y/X} = K_Y + \Delta_Y - f^*(K_X + \Delta).$$

Remark 13 *Our definition of the relative canonical $\mathbb{R}$-divisor is different from the one in [6]. In this paper, the relative canonical $\mathbb{R}$-divisor is defined as $K_{Y/X} = K_Y + f^*(-K_X)$. And $K_Y - f^*K_X$ is denoted by $K^-_{Y/X}$. It can be shown that, with this notation, $K_{Y/X} \geqslant K^-_{Y/X}$. But they are not equal in general. See [6, Example 3.4].*

For every integer $m \geqslant 1$, the m-th limiting log discrepancy $\mathbb{Q}$-divisor $A_{m,Y/X}$ of Y over X is

$$A_{m,Y/X} = K_Y + E_f - \frac{1}{m} f^\natural(mK_X),$$

where E_f is the reduced exceptional divisor of f. The log discrepancy $\mathbb{R}$-divisor $A_{Y/X}$ of Y over X is

$$A_{Y/X} = K_Y + E_f - f^*K_X.$$

The log discrepancy $\mathbb{Q}$-divisor of (Y, Δ_Y) over (X, Δ) is given by

$$A^{\Delta}_{Y/X} = K_Y + \Delta_Y + E_f - f^*(K_X + \Delta).$$

Consider a pair $(X, I = \sum a_k \cdot \mathcal{J}_k)$ where $\mathcal{J}_k$ are nonzero fractional ideal sheaves on X and a_k are real numbers. A log resolution of (X, I) is a proper birational morphism $f : Y \to X$ from a smooth variety Y such that for every k the sheaf $\mathcal{J}_k \cdot \mathcal{O}_Y$ is the invertible sheaf corresponding to a divisor E_k on Y, the exceptional locus $\mathrm{Ex}(f)$ of f is also a divisor, and $\mathrm{Ex}(f) \cup E$ has simple normal crossing, where $E = \bigcup \mathrm{Supp}(E_k)$. If Δ is a boundary on X, then a log resolution of the log pair $((X, \Delta), I)$ is given by a log resolution $f : Y \to X$

of (X, I) such that $\mathrm{Ex}(f) \cup E \cup \mathrm{Supp}(f^*(K_X + \Delta))$ has simple normal crossings. The log resolution always exists (see [6, Theorem 4.2]).

Let X be a normal variety, and fix an integer $m \geqslant 2$. Given a log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$, a boundary Δ on X is said to be m-compatible for X with respect to f if:

1. $m\Delta$ is integral and $\lfloor \Delta \rfloor = 0$,
2. f is a log resolution for the log pair $((X, \Delta); \mathcal{O}_X(-mK_X))$, and
3. $K^{\Delta}_{Y/X} = K_{m,Y/X}$.

Theorem 14 *For any normal variety X, any integer $m \geqslant 2$ and any log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$, there exists an m-compatible boundary Δ for X with respect to f.*

Proof. The idea of the proof is: Let H be a sufficiently ample divisor on X. We pick D to be a general element in the linear series $|-mK_X + mH|$ and Δ to be $\frac{1}{m}D$. Then Δ is m-compatible. For details, see [6, Theorem 5.4]. ■

2.5 BdFF's Non-Log-Canonical Volume

In [29], Wahl studies normal surfaces with noninvertible endomorphisms and defines the local volume for normal surface singularities. In [3], Boucksom, de Fernex and Favre generalize Wahl's volume to higher dimensions using Shokurov's b-divisors. We recall some basic definitions and properties in this section.

2.5.1 Shokurov's b-divisors

Let X be a normal variety. The set of all proper birational morphisms $\pi : X_\pi \to X$ from a normal variety X_π modulo isomorphism is (partially) ordered by $\pi' \geqslant \pi$ if and only if π' factors through π, and any two proper birational morphisms can be dominated by a third one. The Riemann-Zariski space X is defined as the projective limit, $\mathcal{X} = \varprojlim_\pi X_\pi$. The group of Weil b-divisors over X is defined as $\mathrm{Div}(\mathcal{X}) = \varprojlim_\pi \mathrm{Div}(X_\pi)$, where $\mathrm{Div}(X_\pi)$ denotes the group of Weil divisors on X_π and the limit is taken with respect to the pushforwards. The group of Cartier b-divisors over X is defined as $\mathrm{CDiv}(\mathcal{X}) = \varinjlim_\pi \mathrm{CDiv}(X_\pi)$, where $\mathrm{CDiv}(X_\pi)$ denotes the group of Cartier divisors on X_π and the limit is taken with respect to the pullbacks. An element in $\mathrm{Div}_\mathbb{R}(\mathcal{X}) = \mathrm{Div}(\mathcal{X}) \otimes \mathbb{R}$ (resp. $\mathrm{CDiv}_\mathbb{R}(\mathcal{X}) = \mathrm{CDiv}(\mathcal{X}) \otimes \mathbb{R}$) will be called an $\mathbb{R}$-Weil b-divisor (resp. $\mathbb{R}$-Cartier b-divisor), and similarly with $\mathbb{Q}$ in place of $\mathbb{R}$. Clearly, a Weil b-divisor W over X consists of a family of Weil divisors $W_\pi \in \operatorname{Div}(X_\pi)$ that are compatible under pushforward. We say that W_π is the trace of W on the model X_π. Let C be a Cartier b-divisor. We say that $\pi : X_\pi \to X$ is a determination of C, if C can be obtained by pulling back C_π to models dominating π and pushing forward to other models, in which case we denote $C = \overline{C_\pi}$.

Let Z and W be two $\mathbb{R}$-Weil b-divisors over X. We say that $Z \leqslant W$, if for any model $\pi : X_\pi \to X$ we have $Z_\pi \leqslant W_\pi$. We say an $\mathbb{R}$-Cartier b-divisor is relatively nef over X, if its trace is relatively nef on one (hence any sufficiently high) determination. An $\mathbb{R}$-Weil b-divisor W is relatively nef over X if and only if there is a sequence of relatively nef $\mathbb{R}$-Cartier b-divisors over X such that the traces converge to the trace of W in the numerical class over X on each model. Under the setting of relatively nef $\mathbb{R}$-Weil b-divisors, Boucksom, de Fernex and Favre generalized the Negativity Lemma as follows [3, Proposition 2.12]:

Lemma 15 *Let W be a relatively nef $\mathbb{R}$-Weil b-divisor over X. Let $\pi : X_\pi \to X$ and $\pi' : X_{\pi'} \to X$ be two models over X such that π' factors through π via $\rho : X_{\pi'} \to X_\pi$. Then $W_{\pi'} \leqslant -\rho^*(-W_\pi)$.*

Let $C_1, \ldots, C_n$ be $\mathbb{R}$-Cartier b-divisors, where $n = \dim X$. Let f be a common determination. It is clear that the intersection number $C_{1,f} \cdot \ldots \cdot C_{n,f}$ is independent of the choice of f by the projection formula. We define $C_1 \cdot \ldots \cdot C_n$ to be the above intersection number. If $W_1, \ldots, W_n$ are relatively nef $\mathbb{R}$-Weil b-divisors over X, we define

$$W_1 \cdot \ldots \cdot W_n = \inf(C_1 \cdot \ldots \cdot C_n) \in [-\infty, \infty),$$

where the infimum is taken over all relatively nef $\mathbb{R}$-Cartier b-divisors C_i over X such that $C_i \geqslant W_i$ for each i. It is obvious that the intersection number is monotonic in the sense that, if $W_i \leqslant W_i'$ for each i, then $W_1 \cdot \ldots \cdot W_n \leqslant W_1' \cdot \ldots \cdot W_n'$. For further properties of the intersection number, we refer to Section 4.3 and Appendix A of [3].

Given a canonical divisor K_X on X, there is a unique canonical divisor K_{X_π} for each model $\pi : X_\pi \to X$ with the property that $\pi_* K_{X_\pi} = K_X$. Hence, a choice of K_X determines a canonical b-divisor $K_{\mathcal{X}}$ over X. The log discrepancy b-divisor is defined as

$$A_{\mathcal{X}/X} = K_{\mathcal{X}} + E_{\mathcal{X}/X} + \operatorname{Env}_X(-K_X),$$

where the trace of $E_{\mathcal{X}/X}$ in any model π is equal to the reduced exceptional divisor E_π over X. It is clear that the trace of $A_{\mathcal{X}/X}$ on a model $\pi : X_\pi \to X$ is $A_{X_\pi/X}$. Similarly, for every integer $m \geqslant 1$, we define the m-th limiting log discrepancy b-divisor $A_{m,\mathcal{X}/X}$ to be a

$\mathbb{Q}$-Weil b-divisor whose trace on a model $\pi : X_\pi \to X$ is $A_{m,X_\pi/X}$. It is easy to check that $A_{m,\mathcal{X}/X} \leqslant A_{\mathcal{X}/X}$ and $A_{\mathcal{X}/X}$ is the limsup of $A_{m,\mathcal{X}/X}$.

2.5.2 Nef envelopes

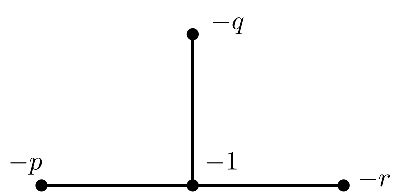
The motivation of nef envelopes traces back to the relative Zariski decomposition of surfaces. Let X be a normal surface and $f : Y \to X$ be its minimal resolution. We may consider the relative Zariski decomposition

$$A_{Y/X} = P + N,$$

where P is f-nef and N is effective and f-exceptional. In particular, P is the largest f-nef divisor such that $P \leqslant A_{Y/X}$. In [29], Wahl defined the volume for normal surface singularity as

$$\mathrm{Vol}(X) = -P^2.$$

Example 16 *Consider the $D_{p,q,r}$-singularity, where $2 \leqslant p \leqslant q \leqslant r$ and $1/p+1/q+1/r < 1$, with resolution dual graph:*



We call the exceptional divisors E_1, E_2, E_3 and E_0, respectively, with self intersections $-p$, $-q$, $-r$ and -1. Then $A_{Y/X} = -E_0$,

$$P = -E_0 - (1/p)E_1 - (1/q)E_2 - (1/r)E_3$$

and

$$\mathrm{Vol}(X) = -P^2 = 1 - 1/p - 1/q - 1/r.$$

In particular, when $(p, q, r) = (2, 3, 7)$, then

$$\mathrm{Vol}(X) = \frac{1}{42}$$

reaches its minimal positive value.

In higher dimensions, the nef envelope $\mathrm{Env}_X(D)$ of an $\mathbb{R}$-divisor D on X is an $\mathbb{R}$-Weil b-divisor over X whose trace on a model $\pi : X_\pi \to X$ is $-\pi^*(-D)$. We refer to [3, Section 2] for further discussions. If D is $\mathbb{Q}$-Cartier, then $\mathrm{Env}_X(D)$ is the $\mathbb{Q}$-Cartier b-divisor $\overline{D}$.

The nef envelope $\operatorname{Env}_{\mathcal{X}}(W)$ of an $\mathbb{R}$-Weil b-divisor W over X is the largest relatively nef $\mathbb{R}$-Weil b-divisor Z over X such that $Z \leqslant W$. It is well-defined by [3, Proposition 2.15]. It is clear that if $W_1 \leqslant W_2$, then $\operatorname{Env}_{\mathcal{X}}(W_1) \leqslant \operatorname{Env}_{\mathcal{X}}(W_2)$.

The volume of the singularity on X is defined by

$$\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = -\operatorname{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})^n.$$

It is shown in [3] that if X has isolated singularity, then $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0)$ is a well-defined non-negative finite real number.

CHAPTER 3

PLURI-CANONICAL MAPS

In this chapter, we will prove the following theorem.

Theorem 17 *Let X be a smooth projective variety of general type over an algebraically closed field k of characteristic $p > 0$. If X has maximal Albanese dimension and the Albanese map is separable, then $|4K_X|$ induces a birational map.*

3.1 Vanishing Theorems

In order to apply Proposition 7, we need to construct a sheaf from the pluri-canonical line bundle and prove the vanishing of higher cohomologies. In characteristic 0, the vanishing can be achieved by Kawamata-Viehweg vanishing. However, in positive characteristic, Kawamata-Viehweg vanishing is known to fail. We will use test ideals and Serre vanishing instead.

Suppose $f : X \to A$ is a nontrivial morphism where X is a smooth variety of general type over an algebraic closed field k of characteristic $p > 0$ and A is an abelian variety. Let K_X be a canonical divisor. Since K_X is big, by Kodaira's Lemma (see [18, Proposition 2.2.6]), we may write $K_X \sim_{\mathbb{Q}} H + E$ where H is an ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor. Let $\Delta = (1-\epsilon)K_X + \epsilon E$, where $\epsilon \in \mathbb{Q}$ and $0 < \epsilon < 1$. Fix a positive integer l such that $l\Delta$ is Cartier. Although Δ is not necessarily effective, since K_X is big and E is effective, we have the Iitaka dimension $\kappa(X, l\Delta) \geqslant 0$. For any positive integer r, let $\mathcal{F}_r = \mathcal{O}_X((r+1)K_X) \otimes \tau(\|r\Delta\|)$.

Let $\mathfrak{a}_{\mathfrak{m}}$ be the base ideal of the linear series $|ml\Delta|$. By the definition of the asymptotic test ideal, we can fix a positive integer m' sufficiently large and divisible such that $\tau(\|r\Delta\|) = \tau(\mathfrak{a}_\bullet^{r/l}) = \tau(\mathfrak{a}_{m'}^{r/m'l})$. We may assume $m' = rm$ for some positive integer m. Then $\tau(\|r\Delta\|) = \tau(\mathfrak{a}_{rm}^{1/ml})$. For every $e \gg 0$, we have $\tau(\mathfrak{a}_{rm}^{1/ml}) = \left(\mathfrak{a}_{rm}^{\lceil p^e/ml \rceil}\right)^{[1/p^e]}$. Hence, the iterated trace map Tr^e gives a surjection

$$\mathrm{Tr}^e : F^e_*(\mathfrak{a}_{rm}^{\lceil p^e/ml \rceil} \cdot \mathcal{O}_X(K_X)) \to \tau(\|r\Delta\|) \cdot \mathcal{O}_X(K_X).$$

Tensoring with $\mathcal{O}_X(rK_X)$, we have a surjection

$$F^e_*(\mathfrak{a}_{rm}^{\lceil p^e/ml\rceil} \cdot \mathcal{O}_X((rp^e+1)K_X)) \to \mathcal{F}_r.$$

Let $\widetilde{\mathcal{F}_{r,e}} = \mathfrak{a}_{rm}^{\lceil p^e/ml\rceil} \cdot \mathcal{O}_X((rp^e+1)K_X)$. Then the surjection above is $F^e_*\widetilde{\mathcal{F}_{r,e}} \to \mathcal{F}_r$. Since $\mathfrak{a}_{rm}$ is the base ideal of $|rml\Delta|$, the evaluation gives a surjection

$$H^0(X, \mathcal{O}_X(rml\Delta)) \otimes \mathcal{O}_X(-rml\Delta) \to \mathfrak{a}_{rm},$$

hence a surjection

$$V_{r,e} \otimes \mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta) \to \mathfrak{a}_{rm}^{\lceil p^e/ml\rceil},$$

where $V_{r,e} = \mathrm{Sym}^{\lceil p^e/ml\rceil} H^0(X, \mathcal{O}_X(rml\Delta))$. Tensoring with $\mathcal{O}_X((rp^e+1)K_X)$, we have a surjection

$$\mathcal{F}_{r,e} = V_{r,e} \otimes \mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X) \to \widetilde{\mathcal{F}_{r,e}}.$$

Pushing forward by the Frobenius and combining with $F^e_*\widetilde{\mathcal{F}_{r,e}} \to \mathcal{F}_r$, we obtain a surjection $F^e_*\mathcal{F}_{r,e} \to \mathcal{F}_r$ since F^e is affine.

Lemma 18 *Fix $r > 0$. Then $R^if_*(F^e_*\mathcal{F}_{r,e}) = 0$ for all $i > 0$ and all e large enough.*

Proof. First, we prove that $R^if_*\mathcal{F}_{r,e} = 0$ for all $i > 0$ and all e large enough. Since $V_{r,e}$ is a vector space over k, we only need to show that

$$R^if_*\mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X) = 0.$$

But

$$\begin{aligned} & -rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X \\ = & -rmls\Delta + (rmls - rt + 1)K_X \\ = & (1-rt)K_X + rmls(K_X - \Delta), \end{aligned}$$

where $s = \lceil p^e/ml\rceil$ and $0 \leqslant t = mls - p^e < ml$. Noticing that $K_X - \Delta \sim_{\mathbb{Q}} \epsilon H$ which is ample, we may apply Serre vanishing. For each value of $t \in [0, ml-1]$, we have $R^if_*\mathcal{O}_X((1-rt)K_X + rmls(K_X - \Delta)) = 0$ for all s large enough, i.e., all e large enough. Thus $R^if_*\mathcal{F}_{r,e} = 0$.

Now, since F^e is exact and commutes with f, we have

$$R^if_*(F^e_*\mathcal{F}_{r,e}) = R^i(f \circ F^e)_*\mathcal{F}_{r,e} = R^i(F^e \circ f)_*\mathcal{F}_{r,e} = F^e_*(R^if_*\mathcal{F}_{r,e}) = 0$$

for all $i > 0$ and all e large enough. ∎

Lemma 19 *Fix $r > 0$. There is an integer $M > 0$ such that $H^i(A, f_*(F^e_*\mathcal{F}_{r,e}) \otimes P) = 0$ for all $i > 0$, $e > M$ and $P \in Pic^0(A)$.*

Proof. The proof is similar to that of Lemma 18. By Lemma 18 and the projection formula, $R^i f_*(F^e_*\mathcal{F}_{r,e} \otimes f^*P) = 0$ for all $i > 0$ and e large enough. Hence, by a spectral sequence argument, it suffices to prove that $H^i(X, F^e_*\mathcal{F}_{r,e} \otimes f^*P) = 0$ or equivalently that $H^i(X, \mathcal{F}_{r,e} \otimes f^*P^{\otimes p^e}) = 0$. We only need to show that

$$H^i(X, \mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X) \otimes f^*P^{\otimes p^e}) = 0.$$

Assume that $s = \lceil p^e/ml\rceil$ and $0 \leqslant t = mls - p^e < ml$. Since $K_X - \Delta \sim_{\mathbb{Q}} \epsilon H$ is ample, by Fujita vanishing, for each value of t, there is an $M_t > 0$ such that for all $e > M_t$ and all nef line bundles $\mathcal{N}$ on X, we have

$$H^i(X, \mathcal{O}_X((1-rt)K_X + rmls(K_X - \Delta)) \otimes \mathcal{N}) = 0.$$

Let $M = \max\{M_t\}$, then

$$H^i(X, \mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X) \otimes \mathcal{N}) = 0$$

for all $e > M$ and all nef line bundles $\mathcal{N}$. In particular, we can take $\mathcal{N} = f^*P^{\otimes p^e}$. The lemma follows. ∎

3.2 Notations

Since the notations used in the proof of Theorem 17 are very complicated, we will fix all the notations in this section.

Suppose A is the Albanese variety and $f : X \to A$ is the Albanese map. Since f is generically finite, there is an open subset U of A such that f is finite over U. And since f is separable, we can fix a canonical divisor $K_X = f^*K_A + R_f = R_f \geqslant 0$. We define

Δ	$= (1-\epsilon)K_X + \epsilon E$ where $K_X \sim_{\mathbb{Q}} H + E$, H is ample and E is effective
l	positive integer such that $l\Delta$ is Cartier
$\mathfrak{a}_m$	base ideal of $\lvert ml\Delta\rvert$
$\mathcal{F}_r$	$= \mathcal{O}_X((r+1)K_X) \otimes \tau(\lVert r\Delta\rVert)$

We fix a positive integer m such that $\tau(\lVert r\Delta\rVert) = \tau(\mathfrak{a}_{rm}^{1/ml})$ for $r = 1, 2, 3$ and define

$\widetilde{\mathcal{F}_{r,e}}$	$= \mathfrak{a}_{rm}^{\lceil p^e/ml\rceil} \cdot \mathcal{O}_X((rp^e+1)K_X)$
$\mathcal{F}_{r,e}$	$= V_{r,e} \otimes \mathcal{O}_X(-rml\lceil p^e/ml\rceil\Delta + (rp^e+1)K_X)$
$\widetilde{\mathcal{F}_{1,e}}^-$	$= \mathfrak{a}_m^{\lceil p^e/ml\rceil} \cdot \mathcal{O}_X(p^eK_X) = \widetilde{\mathcal{F}_{1,e}} \otimes \mathcal{O}_X(-K_X)$
$\mathcal{F}_{1,e}^-$	$= V_{1,e} \otimes \mathcal{O}_X(-ml\lceil p^e/ml\rceil\Delta + p^eK_X) = \mathcal{F}_{1,e} \otimes \mathcal{O}_X(-K_X)$.

where $V_{r,e} = \mathrm{Sym}^{\lceil p^e/ml\rceil} H^0(X, \mathcal{O}_X(rml\Delta))$.

Lemma 20 *For $e \gg 0$, we have $R^i f_*(F^e_* \mathcal{F}^-_{1,e}) = 0$ and $H^i(A, f_*(F^e_* \mathcal{F}^-_{1,e}) \otimes P) = 0$ for all $i > 0$ and $P \in \operatorname{Pic}^0(A)$.*

Proof. The lemma follows immediately from the proofs of Lemma 18 and 19. ■

We fix an integer $e \gg 0$ such that $H^i(A, f_*\mathcal{G} \otimes P) = 0$ holds for all $i > 0$ and all $\mathcal{G} \in \{\mathcal{F}_{1,e}, \mathcal{F}_{2,e}, \mathcal{F}_{3,e}, \mathcal{F}^-_{1,e}\}$. By a general point $a \in A$, we mean a point $a \in U$. By a general point $x \in X$, we mean a point $x \in f^{-1}(U)$ such that x is not in the co-supports of $\mathfrak{a}_m^{\lceil p^e/ml \rceil}$ or $\tau(\|3\Delta\|)$ (hence, not in the co-supports of $\tau(\|2\Delta\|)$ or $\tau(\|\Delta\|)$ by Remark 21) and that x is not in the support of $K_X = R_f$. (It is not hard to see that x is not in the co-support of $\mathfrak{a}_m$ is equivalent to that x is not in the co-support of $\mathfrak{a}_m^{\lceil p^e/ml \rceil}$ and implies that x is not in the co-support of $\tau(\|3\Delta\|)$.)

Fix a positive integer m such that $\tau(\|r\Delta\|) = \tau(\mathfrak{a}_{rm}^{1/ml})$. Let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_X$. In our applications, $\mathcal{I} = \mathcal{O}_X$ or $\mathcal{I} = \mathcal{I}_x$, where $\mathcal{I}_x$ is the maximal ideal of closed point. The composition of the two surjections, $F^e_*\mathcal{F}_{r,e} \to \mathcal{F}_r$ and $\mathcal{F}_r \to \mathcal{F}_r \otimes \mathcal{O}_X/\mathcal{I}$, is still surjective. We define $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$ to be the kernel of this composition. Then $(F^e_*\mathcal{F}_{r,e})_{\mathcal{O}_X} = F^e_*\mathcal{F}_{r,e}$. Assuming that

$$\text{the intersection of the co-supports of } \tau(\|r\Delta\|) \text{ and } \mathcal{I} \text{ is empty,} \tag{$(*)_r$}$$

since the composition $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to F^e_*\mathcal{F}_{r,e} \to \mathcal{F}_r \to \mathcal{F}_r \otimes \mathcal{O}_X/\mathcal{I}$ is 0, it factors through the kernel of $\mathcal{F}_r \to \mathcal{F}_r \otimes \mathcal{O}_X/\mathcal{I}$, which is $\mathcal{F}_r \otimes \mathcal{I}$. We have a map $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to \mathcal{F}_r \otimes \mathcal{I}$, and by the 5-lemma, it is surjective. This is summarized in the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \arrow[r] \arrow[d] & F^e_*\mathcal{F}_{r,e} \arrow[r] \arrow[d] & \mathcal{F}_r \otimes \mathcal{O}_X/\mathcal{I} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F}_r \otimes \mathcal{I} \arrow[r] & \mathcal{F}_r \arrow[r] & \mathcal{F}_r \otimes \mathcal{O}_X/\mathcal{I} \arrow[r] & 0
\end{tikzcd}$$

Remark 21 *The condition $(*)_r$ is true if $\mathcal{I} = \mathcal{O}_X$ or $\mathcal{I} = \mathcal{I}_x$ where x is not in the co-support of $\tau(\|r\Delta\|)$. And $(*)_r$ implies $(*)_s$ if $r \geqslant s$, since $\tau(\|r\Delta\|) \subseteq \tau(\|s\Delta\|)$.*

Suppose that x is a point in X such that x is not in the co-support of the ideals $\tau(\|r\Delta\|)$ or $\mathcal{I}$. Then the restriction to the point x gives a surjection $\mathcal{F}_r \otimes \mathcal{I} \to \mathcal{F}_r \otimes k(x) \cong k(x)$. Hence, a surjection $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to \mathcal{F}_r \otimes k(x)$. Let $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x}$ be the kernel. Thus, we have the following exact sequence

$$0 \to (F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to \mathcal{F}_r \otimes k(x) \to 0.$$

3.3 Proof of Theorem 17

Let $f : X \to A$ be a nontrivial separable morphism where A is an abelian variety.

Theorem 22 *Fix $e > 0$ and r a positive integer. Let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_X$ satisfying $(*)_r$. Suppose that x is a point in X such that*

1. *x is not in the co-support of $\tau(\|r\Delta\|)$ or $\mathcal{I}$,*

2. *$f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \neq f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$,*

3. *$H^i(A, f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \otimes P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$.*

Then the homomorphism $H^0(X, (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \otimes f^*P) \to H^0(X, \mathcal{F}_r \otimes \mathcal{I} \otimes f^*P \otimes k(x)) \cong k(x)$ induced by $\phi_{r,e,\mathcal{I}}$ is surjective for general $P \in \mathrm{Pic}^0(A)$. Moreover, x is not a base point of $\mathcal{F}_r \otimes \mathcal{I} \otimes f^*P$ for general $P \in \mathrm{Pic}^0(A)$.

Proof. Pushing forward the exact sequence

$$0 \to (F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to \mathcal{F}_r \otimes k(x) \to 0,$$

we have

$$0 \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to f_*(\mathcal{F}_r \otimes k(x)) \to R^1f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to \cdots .$$

Let $a = f(x)$. Since f is separable, we have that a is reduced, hence $f_*(\mathcal{F}_r \otimes k(x)) \cong k(a)$. By assumption, $f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$ is not an isomorphism, which implies that the kernel of $k(a) \to R^1f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x}$ is not 0. But the kernel is a sub-sheaf of $k(a)$ which has no nonzero sub-sheaf other than itself. Hence, the kernel is $k(a)$ and we have an exact sequence

$$0 \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to k(a) \to 0.$$

Applying Proposition 7 to the surjection $f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to k(a)$, we have the surjection $H^0(f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \otimes P) \to k(a)$ for general $P \in \mathrm{Pic}^0(A)$. Hence, the theorem follows. For the moreover part, noticing that the surjection factors through $H^0(\mathcal{F}_r \otimes \mathcal{I} \otimes f^*P)$, we have that the induced homomorphism $H^0(\mathcal{F}_r \otimes \mathcal{I} \otimes f^*P) \to k(x)$ is also surjective. ■

The following corollary is useful in the case of the maximal Albanese dimension.

Corollary 23 *Suppose f is finite over an open subset U in A. Fix $e > 0$ and r a positive integer. Let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_X$ satisfying $(*)_r$. Suppose that x is a point in X such that*

1. x is not in the co-support of $\tau(\|r\Delta\|)$ or $\mathcal{I}$,

2. $a = f(x) \in U$,

3. $H^i(A, f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \otimes P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$.

Then the conclusion of Theorem 22 still holds.

Proof. By Theorem 22, we only need to show that $f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \neq f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$. Recall that we have an exact sequence

$$0 \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to f_*(\mathcal{F}_r \otimes k(x)) \to R^1 f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} \to \cdots .$$

If $f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x} = f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}}$, we have that the map $f_*(\mathcal{F}_r \otimes k(x)) \to R^1 f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x}$ is nonzero. So the stalk of $R^1 f_*(F^e_*\mathcal{F}_{r,e})_{\mathcal{I},x}$ at a is nonzero. But as f is finite over a, the higher direct images are 0 at a, a contradiction. ■

Remark 24 *It is easy to see that the proof of Theorem 22 and Corollary 23 not only works for the surjection $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}} \to \mathcal{F}_r \otimes k(x)$, but any surjection to the trivial skyscraper sheaf satisfying the vanishing condition (3). We will use this variant version repeatedly in the proof of Theorem 17.*

Theorem 22 also gives information on the base locus of $\mathcal{O}_X(2K_X) \otimes f^*P$ for general $P \in \mathrm{Pic}^0(A)$.

Corollary 25 *Fix $e > M$ as in Lemma 19. Suppose that x is a point in X such that*

1. x is not in the co-support of $\tau(\|\Delta\|)$,

2. $f_*(F^e_*\mathcal{F}_{1,e})_{\mathcal{O}_X,x} \neq f_*(F^e_*\mathcal{F}_{1,e})$.

Then x is not a base point of $\mathcal{F}_1 \otimes f^*P$ for general $P \in \mathrm{Pic}^0(A)$. Hence, x is not a base point of $\mathcal{O}_X(2K_X) \otimes f^*P$ for general $P \in \mathrm{Pic}^0(A)$.

Proof. The first part of the corollary follows directly from Theorem 22 and Lemma 19. The second part follows from the facts that $\mathcal{F}_1 = \mathcal{O}_X(2K_X) \otimes \tau(\|\Delta\|)$ and $\tau(\|\Delta\|)$ is an ideal. ■

We are ready to prove the main result.

Proof of Theorem 17: Our strategy is: First, by Theorem 22, we have that x is not a base point of $\mathcal{F}_1 \otimes f^*P$ for general $P \in \mathrm{Pic}^0(A)$. Then, by comparing $\mathcal{F}_1$ and $\mathcal{F}_2$ via $\mathcal{F}^-_{1,e}$, we show that x is not a base point of $\mathcal{F}_2 \otimes f^*P$ for all $P \in \mathrm{Pic}^0(A)$. Using this fact, we show that $\mathcal{F}_2 \otimes f^*P$ separates points for general $P \in \mathrm{Pic}^0(A)$. Finally, by comparing $\mathcal{F}_2$ and $\mathcal{F}_3$ via $\mathcal{F}_{1,e}^-$, we have that $\mathcal{F}_3 \otimes f^*P$ separates points for all $P \in \mathrm{Pic}^0(A)$. Hence, so does $\mathcal{F}_3$. Following the same idea, we show that $\mathcal{F}_3$ separates tangent vectors.

Step 1. By Lemma 19 and Corollary 23 with $r = 1$ and $\mathcal{I} = \mathcal{O}_X$, we have that for general $x \in X$, the homomorphism

$$H^0(X, F_*^e\mathcal{F}_{1,e} \otimes f^*P) \to H^0(X, \mathcal{F}_1 \otimes f^*P \otimes k(x)) \cong k(x)$$

is surjective for general $P \in \mathrm{Pic}^0(A)$.

Step 2. We show that for general $x \in X$, the homomorphism

$$H^0(X, F_*^e\mathcal{F}_{2,e} \otimes f^*Q) \to H^0(X, \mathcal{F}_2 \otimes f^*Q \otimes k(x)) \cong k(x)$$

is surjective for all $Q \in \mathrm{Pic}^0(A)$.

Let us give a quick explanation of the idea in this step first. We can pick $P \in \mathrm{Pic}^0(A)$ such that both P and $Q \otimes P^\vee$ are general. We have already shown that there is a global section of $F_*^e\mathcal{F}_{1,e} \otimes f^*P$ which induces a global section of $\mathcal{F}_1 \otimes f^*P$ not vanishing at x. Notice that the difference between $\mathcal{F}_1 \otimes f^*P$ and $\mathcal{F}_2 \otimes f^*Q$ near a general point x is $\mathcal{O}_X(K_X) \otimes f^*(Q \otimes P^\vee)$. If we can find a global section of $\mathcal{O}_X(K_X)$ not vanishing at x, we can obtain a global section of $\mathcal{F}_2 \otimes f^*Q$ not vanishing at x. This can be done as K_X is effective. But, unfortunately, $\mathcal{O}_X(K_X)$ does not behave well globally with the Frobenius, $\widetilde{\mathcal{F}_{r,e}}$ and $\mathcal{F}_{r,e}$. We have to introduce $\widetilde{\mathcal{F}_{1,e}}^-$ and $\mathcal{F}_{1,e}^-$ as the bridge from $\widetilde{\mathcal{F}_{1,e}}$ to $\widetilde{\mathcal{F}_{2,e}}$ and $\mathcal{F}_{1,e}$ to $\mathcal{F}_{2,e}$, respectively. The induced map $F_*^e\mathcal{F}_{1,e} \to F_*^e\mathcal{F}_{2,e}$ is commutative with $\mathcal{F}_1 \to \mathcal{O}_X(K_X) \otimes \mathcal{F}_1 \cong \mathcal{F}_2$ near a general point x by the projection formula. Hence, we view $F_*^e\mathcal{F}_{1,e}^-$ as giving a homomorphism $\mathcal{F}_1 \otimes k(x) \to \mathcal{F}_2 \otimes k(x)$. We only need to show that there is a global section of $F_*^e\mathcal{F}_{1,e}^-$ inducing a nonzero homomorphism $\mathcal{F}_1 \otimes k(x) \to \mathcal{F}_2 \otimes k(x)$. Here is the detailed proof.

Since $\mathfrak{a}_m^{\lceil p^e/ml \rceil}$ is an ideal, we have an inclusion $\widetilde{\mathcal{F}_{1,e}}^- \to \mathcal{O}_X(p^eK_X)$. Tensoring with the vector bundle $\mathcal{F}_{1,e}$, we have an inclusion

$$\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e} \to \mathcal{O}_X(p^eK_X) \otimes \mathcal{F}_{1,e},$$

whose cokernel is supported on the co-support of $\mathfrak{a}_m^{\lceil p^e/ml \rceil}$. Pushing forward by the Frobenius, we get another inclusion

$$F_*^e(\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e}) \to F_*^e(\mathcal{O}_X(p^eK_X) \otimes \mathcal{F}_{1,e}) \cong \mathcal{O}_X(K_X) \otimes F_*^e(\mathcal{F}_{1,e}),$$

whose cokernel is still supported on the co-support of $\mathfrak{a}_m^{\lceil p^e/ml \rceil}$. Hence, the induced morphism

$$\alpha : F^e_*(\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e}) \otimes k(x) \to \mathcal{O}_X(K_X) \otimes F^e_*(\mathcal{F}_{1,e}) \otimes k(x)$$

is an isomorphism providing that x is general. Since $\mathfrak{a}_m^2 \subseteq \mathfrak{a}_{2m}$, we have a morphism $\widetilde{\mathcal{F}_{1,e}}^- \otimes \widetilde{\mathcal{F}_{1,e}} \to \widetilde{\mathcal{F}_{2,e}}$. Combining with the morphism $\mathcal{F}_{1,e} \to \widetilde{\mathcal{F}_{1,e}}$, we have that

$$\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e} \to \widetilde{\mathcal{F}_{1,e}}^- \otimes \widetilde{\mathcal{F}_{1,e}} \to \widetilde{\mathcal{F}_{2,e}}.$$

On the other hand, since $\tau(\|\Delta\|)$ and $\tau(\|2\Delta\|)$ are ideals, the induced inclusions $\mathcal{F}_1 \otimes k(x) \to \mathcal{O}_X(2K_X) \otimes k(x)$ and $\mathcal{F}_2 \otimes k(x) \to \mathcal{O}_X(3K_X) \otimes k(x)$ are both isomorphisms providing that x is general. Hence, there is a morphism

$$\mathcal{F}_1 \otimes k(x) \to \mathcal{O}_X(K_X) \otimes \mathcal{F}_1 \otimes k(x) \cong \mathcal{O}_X(3K_X) \otimes k(x) \cong \mathcal{F}_2 \otimes k(x).$$

Combining the discussion above and the trace maps $F^e_*\mathcal{F}_{r,e} \to F^e_*\widetilde{\mathcal{F}_{r,e}} \to \mathcal{F}_r$, we have the following commutative diagram:

$$\begin{tikzcd}
F^e_*\mathcal{F}_{1,e} \otimes k(x) \arrow[r] \arrow[d, two heads] & \mathcal{O}_X(K_X) \otimes F^e_*\mathcal{F}_{1,e} \otimes k(x) \arrow[r, "\simeq", "\alpha^{-1}"'] \arrow[d, two heads] & F^e_*(\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e}) \otimes k(x) \arrow[r] \arrow[d, two heads] & F^e_*\widetilde{\mathcal{F}_{2,e}} \otimes k(x) \arrow[d, two heads] \\
\mathcal{F}_1 \otimes k(x) \arrow[r, "\simeq"] & \mathcal{O}_X(K_X) \otimes \mathcal{F}_1 \otimes k(x) \arrow[r, equal] & \mathcal{O}_X(K_X) \otimes \mathcal{F}_1 \otimes k(x) \arrow[r, "\simeq"] & \mathcal{F}_2 \otimes k(x).
\end{tikzcd}$$

The surjectivities of the first, second and last vertical maps are induced by the surjectivity of $F^e_*\mathcal{F}_{r,e} \to \mathcal{F}_r$. The third map is the same as the second map.

Noticing that $\mathcal{F}_{1,e}$ is a vector bundle, we have that the morphism $\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e} \to \widetilde{\mathcal{F}_{2,e}}$ is equivalent to a morphism $\widetilde{\mathcal{F}_{1,e}}^- \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_{1,e}, \widetilde{\mathcal{F}_{2,e}})$. We have

$$\begin{aligned}
F^e_*\widetilde{\mathcal{F}_{1,e}}^- &\to F^e_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_{1,e}, \widetilde{\mathcal{F}_{2,e}}) \\
&\to \mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{F}_{1,e}, F^e_*\widetilde{\mathcal{F}_{2,e}}) \\
&\to \mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{F}_{1,e} \otimes k(x), F^e_*\widetilde{\mathcal{F}_{2,e}} \otimes k(x))
\end{aligned}$$

which, by construction, is how $F^e_*\widetilde{\mathcal{F}_{1,e}}^-$ induces the top row of the commutative diagram above. This induces a morphism between $\mathcal{F}_1 \otimes k(x)$ and $\mathcal{F}_2 \otimes k(x)$. Indeed, for any $a \in \mathcal{F}_1 \otimes k(x)$, we have some $b \in F^e_*\mathcal{F}_{1,e} \otimes k(x)$ (maybe not unique) mapped to a by the first vertical arrow in the commutative diagram. Applying the top row induced by $F^e_*\widetilde{\mathcal{F}_{1,e}}^-$ and then the last vertical arrow on b, we get some $c \in \mathcal{F}_2 \otimes k(x)$ which is independent of the choice of b since the diagram commutes. Hence, we have a morphism

$$F^e_*\widetilde{\mathcal{F}_{1,e}}^- \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes k(x), \mathcal{F}_2 \otimes k(x)) \cong k(x).$$

Remark 26 *We should point out that, on any affine open set V in X, the map that we constructed above between $F^e_*\mathcal{F}_{1,e}$ and $F^e_*\widetilde{\mathcal{F}_{2,e}}$ is not*

$$F^e_*\mathcal{F}_{1,e} \to F^e_*\widetilde{\mathcal{F}_{1,e}}^- \otimes F^e_*\mathcal{F}_{1,e} \to F^e_*(\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e}) \to F^e_*\widetilde{\mathcal{F}_{2,e}},$$

where the second map is the natural morphism of pushing forward a tensor product. The map that we constructed is

$$F^e_*\mathcal{F}_{1,e} \to F^e_*(\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e}) \to F^e_*\widetilde{\mathcal{F}_{2,e}},$$

where the first map is by pushing forward $\mathcal{F}_{1,e} \to \widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e}$.

Assuming that x is not in the support of the effective divisor K_X that we fixed as the ramification divisor before, since the bottom row of the commutative diagram are all isomorphisms in this case, the morphism above $F^e_*\widetilde{\mathcal{F}_{1,e}}^- \to k(x)$ is nonzero, and hence surjective. Combining with the surjection $\mathcal{F}^-_{1,e} \to \widetilde{\mathcal{F}_{1,e}}^-$, we have the following surjection:

$$F^e_*\mathcal{F}^-_{1,e} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes k(x), \mathcal{F}_2 \otimes k(x)) \cong k(x).$$

For any $Q \in \mathrm{Pic}^0(A)$, we can pick $P \in \mathrm{Pic}^0(A)$ such that P and $Q \otimes P^\vee$ are both general. Applying Corollary 23 and Remark 24 to the surjection $F^e_*\mathcal{F}^-_{1,e} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes k(x), \mathcal{F}_2 \otimes k(x))$, since $Q \otimes P^\vee$ is general, we get a surjection

$$H^0(X, F^e_*\mathcal{F}^-_{1,e} \otimes f^*(Q \otimes P^\vee)) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_1 \otimes f^*P \otimes k(x), \mathcal{F}_2 \otimes f^*Q \otimes k(x)).$$

Combining with the fact from Step 1, that $H^0(X, F^e_*\mathcal{F}_{1,e} \otimes f^*P) \to H^0(X, \mathcal{F}_1 \otimes f^*P \otimes k(x))$ is surjective, we have a surjection

$$H^0(X, F^e_*\mathcal{F}^-_{1,e} \otimes F^e_*\mathcal{F}_{1,e} \otimes f^*Q) \to H^0(X, \mathcal{F}_2 \otimes f^*Q \otimes k(x)).$$

Notice that there is a natural commutative diagram

$$\begin{tikzcd}
\mathcal{F}^-_{1,e} \otimes \mathcal{F}_{1,e} \arrow[r] \arrow[d] & \mathcal{F}_{2,e} \arrow[dd] \\
\widetilde{\mathcal{F}_{1,e}}^- \otimes \mathcal{F}_{1,e} \arrow[d] & \\
\widetilde{\mathcal{F}_{1,e}}^- \otimes \widetilde{\mathcal{F}_{1,e}} \arrow[r] & \widetilde{\mathcal{F}_{2,e}}
\end{tikzcd}$$

By construction, $F^e_*\mathcal{F}^-_{1,e} \otimes F^e_*\mathcal{F}_{1,e} \to \mathcal{F}_2$ factors through $F^e_*\mathcal{F}_{2,e}$. Therefore, we have that the homomorphism

$$H^0(X, F^e_*\mathcal{F}_{2,e} \otimes f^*Q) \to H^0(X, \mathcal{F}_2 \otimes f^*Q \otimes k(x))$$

is surjective for all $Q \in \mathrm{Pic}^0(A)$.

Step 3. We show that for general $x_1, x_2 \in X$ and general $P \in \mathrm{Pic}^0(A)$, we can find a section in $F^e_*\mathcal{F}_{2,e} \otimes f^*P$ which induces a section in $\mathcal{F}_2 \otimes f^*P$ vanishing at x_1 but not at x_2.

We only need to show that the map

$$H^0(X, (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) \to H^0(X, \mathcal{F}_2 \otimes \mathcal{I}_{x_1} \otimes f^*P \otimes k(x_2)) \cong k(x_2)$$

is surjective. Noticing that x_1 is not in the co-support of $\tau(\|2\Delta\|)$, we have that $\mathcal{I}_{x_1}$ satisfies $(*)_2$. Applying Corollary 23 with $r = 2$ and $\mathcal{I} = \mathcal{I}_{x_1}$, it suffices to check $H^i(A, f_*(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$.

First we show that $R^if_*(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} = 0$ for $i > 0$. Pushing forward the exact sequence

$$0 \to (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \to F^e_*\mathcal{F}_{2,e} \to \mathcal{F}_2 \otimes k(x_1) \to 0$$

gives

$$f_*(F^e_*\mathcal{F}_{2,e}) \to k(a_1) \to R^1f_*(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \to R^1f_*(F^e_*\mathcal{F}_{2,e}) = 0$$

and

$$R^if_*(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \cong R^if_*(F^e_*\mathcal{F}_{2,e}) = 0$$

for all $i \geqslant 2$, where $a_1 = f(x_1)$ and the vanishings follow from Lemma 18. As in the proof of Theorem 22 (notice that $(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} = (F^e_*\mathcal{F}_{2,e})_{\mathcal{O}_X,x_1}$), one sees that $f_*(F^e_*\mathcal{F}_{2,e}) \to k(a_1)$ is surjective, so $R^1f_*(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} = 0$.

Now, by a spectral sequence argument, we only need to show that $H^i(X, (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$. We have the short exact sequence

$$0 \to (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P \to F^e_*\mathcal{F}_{2,e} \otimes f^*P \to \mathcal{F}_2 \otimes f^*P \otimes k(x_1) \to 0.$$

By taking the cohomology, we have

$$H^0(X, F^e_*\mathcal{F}_{2,e} \otimes f^*P) \to k(x_1) \to H^1(X, (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) \to H^1(X, F^e_*\mathcal{F}_{2,e} \otimes f^*P) = 0$$

and

$$H^i(X, (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) \cong H^i(X, F^e_*\mathcal{F}_{2,e} \otimes f^*P) = 0,$$

for all $i \geqslant 2$ where the vanishings follow from Lemma 19. Since by Step 2, $H^0(X, F^e_*\mathcal{F}_{2,e} \otimes f^*P) \to k(x_1)$ is surjective, we have $H^1(X, (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_{x_1}} \otimes f^*P) = 0$.

Step 4. We show that for general $x_1, x_2 \in X$ and all $Q \in \mathrm{Pic}^0(A)$, we can find a section in $F^e_*\mathcal{F}_{3,e} \otimes f^*Q$ which induces a section in $\mathcal{F}_3 \otimes f^*Q$ vanishing at x_1 but not at x_2.

For any general points x_1 and x_2 and any $Q \in \mathrm{Pic}^0(A)$, we may pick $P \in \mathrm{Pic}^0(A)$ such that P and $Q \otimes P^\vee$ are both general. Similar to Step 2, for $i = 1$ or 2, we have the following commutative diagram:

$$\begin{tikzcd}
F^e_*\mathcal{F}_{2,e} \otimes k(x_i) \arrow[r] \arrow[d] & \mathcal{O}_X(K_X) \otimes F^e_*\mathcal{F}_{2,e} \otimes k(x_i) \arrow[r, "\simeq"] \arrow[d] & F^e_*(\widetilde{\mathcal{F}_{1,e}}^{-} \otimes \mathcal{F}_{2,e}) \otimes k(x_i) \arrow[r] \arrow[d] & F^e_*\widetilde{\mathcal{F}_{3,e}} \otimes k(x_i) \arrow[d] \\
\mathcal{F}_2 \otimes k(x_i) \arrow[r, "\simeq"] & \mathcal{O}_X(K_X) \otimes \mathcal{F}_2 \otimes k(x_i) \arrow[r, equal] & \mathcal{O}_X(K_X) \otimes \mathcal{F}_2 \otimes k(x_i) \arrow[r, "\simeq"] & \mathcal{F}_3 \otimes k(x_i).
\end{tikzcd}$$

We have that

$$F^e_*\mathcal{F}^-_{1,e} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2 \otimes k(x_i), \mathcal{F}_3 \otimes k(x_i)) \cong k(x_i)$$

is surjective. We may apply Corollary 23 and Remark 24 and get that

$$H^0(X, F^e_*\mathcal{F}^-_{1,e} \otimes f^*(Q \otimes P^\vee)) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_2 \otimes f^*P \otimes k(x_2), \mathcal{F}_3 \otimes f^*Q \otimes k(x_2))$$

is surjective.

By Step 3, we have a section $s \in H^0(X, F^e_*\mathcal{F}_{2,e} \otimes f^*P)$ restricting to 0 in $\mathcal{F}_2 \otimes f^*P \otimes k(x_1)$ and to nonzero in $\mathcal{F}_2 \otimes f^*P \otimes k(x_2)$. By the discussion above, we have a section $s^- \in H^0(X, F^e_*\mathcal{F}^-_{1,e} \otimes f^*(Q \otimes P^\vee))$ inducing a nonzero homomorphism between $\mathcal{F}_2 \otimes f^*P \otimes k(x_2)$ and $\mathcal{F}_3 \otimes f^*Q \otimes k(x_2)$. Hence, $s^- \otimes s$ gives a section in $H^0(X, F^e_*\mathcal{F}_{3,e} \otimes f^*Q)$ restricting to 0 in $\mathcal{F}_3 \otimes f^*Q \otimes k(x_1)$ and to nonzero in $\mathcal{F}_3 \otimes f^*Q \otimes k(x_2)$.

Step 5. By Step 4, for all $Q \in \mathrm{Pic}^0(A)$, we have a surjection

$$H^0(X, F^e_*\mathcal{F}_{3,e} \otimes f^*Q) \to H^0(X, \mathcal{F}_3 \otimes f^*Q \otimes k(x_1, x_2)),$$

where $k(x_1, x_2)$ is the skyscraper sheaf supported on $\{x_1, x_2\}$. Since this surjection factors through $H^0(X, \mathcal{F}_3 \otimes f^*Q)$, we have that $\mathcal{F}_3 \otimes f^*Q$ separates general points for all $Q \in \mathrm{Pic}^0(A)$.

Step 6. We show that for general $x \in X$, any irreducible length two zero dimensional scheme z with support x and general $P \in \mathrm{Pic}^0(A)$, we can find a section in $(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_x} \otimes f^*P$ which induces a section in $\mathcal{F}_2 \otimes f^*P \otimes \mathcal{I}_x$ not vanishing at z.

Let $r \in \{2, 3\}$ and $\mathcal{I}_z$ be the ideal sheaf of z in X. Since x is not in the co-support of $\tau(\|r\Delta\|)$, the natural map $\mathcal{F}_r \otimes \mathcal{I}_x \to \mathcal{F}_r \otimes \mathcal{I}_x/\mathcal{I}_z$ is surjective with kernel $\mathcal{F}_r \otimes \mathcal{I}_z$. Recall that we have a surjection $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}_x} \to \mathcal{F}_r \otimes \mathcal{I}_x$. Hence, the composition

$$(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}_x} \to \mathcal{F}_r \otimes \mathcal{I}_x \to \mathcal{F}_r \otimes \mathcal{I}_x/\mathcal{I}_z$$

is surjective. We define $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}_x,z}$ as the kernel of the composition above. Since the composition $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}_x,z} \to (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}_x} \to \mathcal{F}_r \otimes \mathcal{I}_x \to \mathcal{F}_r \otimes \mathcal{I}_x/\mathcal{I}_z$ is 0, it factors through $\mathcal{F}_r \otimes \mathcal{I}_z$. By the 5-lemma, the induced map $(F^e_*\mathcal{F}_{r,e})_{\mathcal{I}_x,z} \to \mathcal{F}_r \otimes \mathcal{I}_z$ is surjective. This is summarized in the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}_x,z} & \longrightarrow & (F^e_*\mathcal{F}_{r,e})_{\mathcal{I}_x} & \longrightarrow & \mathcal{F}_r\otimes\mathcal{I}_x/\mathcal{I}_z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{F}_r\otimes\mathcal{I}_z & \longrightarrow & \mathcal{F}_r\otimes\mathcal{I}_x & \longrightarrow & \mathcal{F}_r\otimes\mathcal{I}_x/\mathcal{I}_z & \longrightarrow & 0 \end{array}$$

To show the claim at the beginning of this step, we only need to show that the map

$$H^0(X,(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_x}\otimes f^*P)\to H^0(X,\mathcal{F}_2\otimes\mathcal{I}_x/\mathcal{I}_z\otimes f^*P)\cong k(x)$$

is surjective. Suppose $a=f(x)$ and $t=f(z)$. Since f is separable and x is not in the co-support of $\tau(\|2\Delta\|)$, we have that $f_*(\mathcal{F}_2\otimes\mathcal{I}_x/\mathcal{I}_z)\cong k(a)$ which is the trivial skyscraper sheaf at a. Noticing that $\mathcal{I}_x$ satisfies $(*)_2$, it is not hard to see that the proof of Theorem 22 and Corollary 23 still works for $r=2,\mathcal{I}=\mathcal{I}_x$ and the surjection $(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_x}\to\mathcal{F}_2\otimes\mathcal{I}_x/\mathcal{I}_z\cong k(x)$. The required vanishing $H^i(A,f_*(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_x}\otimes P)=0$ for all $i>0$ and all $P\in\mathrm{Pic}^0(A)$ is shown in Step 3.

Step 7. We show that for general $x\in X$, any irreducible length two zero dimensional scheme z with support x and all $Q\in\mathrm{Pic}^0(A)$, we can find a section in $(F^e_*\mathcal{F}_{3,e})_{\mathcal{I}_x}\otimes f^*Q$ which induces a section in $\mathcal{F}_3\otimes f^*P\otimes\mathcal{I}_x$ not vanishing at z.

Let the kernel of the composition $F^e_*\widetilde{\mathcal{F}_{3,e}}\to\mathcal{F}_3\to\mathcal{F}_3\otimes k(x)$ be $(F^e_*\widetilde{\mathcal{F}_{3,e}})_{\mathcal{I}_x}$. As in Step 2 and Step 4, near a general point x, $F^e_*\widetilde{\mathcal{F}_{1,e}}^-$ induces homomorphisms from $F^e_*\mathcal{F}_{2,e}$ to $F^e_*\widetilde{\mathcal{F}_{3,e}}$ which is commutative with the homomorphisms from $\mathcal{F}_2$ to $\mathcal{F}_3$ induced by $\mathcal{O}_X(K_X)$. Hence, $F^e_*\widetilde{\mathcal{F}_{1,e}}^-$ induces homomorphisms between the kernels of $F^e_*\mathcal{F}_{r,e}\to\mathcal{F}_r\otimes k(x)$, i.e, from $(F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_x}$ to $(F^e_*\widetilde{\mathcal{F}_{3,e}})_{\mathcal{I}_x}$. As the homomorphism induced by $F^e_*\widetilde{\mathcal{F}_{1,e}}^-$ and $\mathcal{O}_X(K_X)$ is commutative, we have the following commutative diagram near a general point x.

$$\begin{array}{ccccc} (F^e_*\mathcal{F}_{2,e})_{\mathcal{I}_x} & \longrightarrow & & \longrightarrow & (F^e_*\widetilde{\mathcal{F}_{3,e}})_{\mathcal{I}_x} \\ \downarrow & & & & \downarrow \\ \mathcal{F}_2\otimes\mathcal{I}_x/\mathcal{I}_z & \longrightarrow & \mathcal{O}_X(K_X)\otimes\mathcal{F}_2\otimes\mathcal{I}_x/\mathcal{I}_z & \xrightarrow{\simeq} & \mathcal{F}_3\otimes\mathcal{I}_x/\mathcal{I}_z \end{array}$$

For any $Q\in\mathrm{Pic}^0(A)$, we may pick $P\in\mathrm{Pic}^0(A)$ such that P and $Q\otimes P^\vee$ are both general. Similar to Step 2 and Step 4, we have that

$$F^e_*\mathcal{F}^-_{1,e}\to F^e_*\widetilde{\mathcal{F}_{1,e}}^-\to\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2\otimes\mathcal{I}_x/\mathcal{I}_z,\mathcal{F}_3\otimes\mathcal{I}_x/\mathcal{I}_z)\cong k(x)$$

is surjective. We may apply Corollary 23 and Remark 24 and get that

$$H^0(X,F^e_*\mathcal{F}^-_{1,e}\otimes f^*(Q\otimes P^\vee))\to\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_2\otimes f^*P\otimes\mathcal{I}_x/\mathcal{I}_z,\mathcal{F}_3\otimes f^*Q\otimes\mathcal{I}_x/\mathcal{I}_z)$$

is surjective.

By Step 6, we have a section $s \in H^0(X, F^e_* \mathcal{F}_{2,e} \otimes f^*P)$ restricting to 0 in $\mathcal{F}_2 \otimes f^*P \otimes \mathcal{O}_X/\mathcal{I}_x$ and whose image in $\mathcal{F}_2 \otimes f^*P \otimes \mathcal{I}_x/\mathcal{I}_z$ does not vanish. By the discussion above, we have a section $s^- \in H^0(X, F^e_* \mathcal{F}^-_{1,e} \otimes f^*(Q \otimes P^\vee))$ inducing a nonzero homomorphism between $\mathcal{F}_2 \otimes f^*P \otimes \mathcal{I}_x/\mathcal{I}_z$ and $\mathcal{F}_3 \otimes f^*Q \otimes \mathcal{I}_x/\mathcal{I}_z$. Hence, $s^- \otimes s$ gives a section in $H^0(X, F^e_* \mathcal{F}_{3,e} \otimes f^*Q)$ restricting to 0 in $\mathcal{F}_3 \otimes f^*Q \otimes \mathcal{O}_X/\mathcal{I}_x$ and whose image in $\mathcal{F}_3 \otimes f^*Q \otimes \mathcal{I}_x/\mathcal{I}_z$ does not vanish.

Step 8. By Step 7, for all $Q \in \mathrm{Pic}^0(A)$, we have a surjection

$$H^0(X, (F^e_* \mathcal{F}_{3,e})_{\mathcal{I}_x} \otimes f^*Q) \to H^0(X, \mathcal{F}_3 \otimes f^*Q \otimes \mathcal{I}_x/\mathcal{I}_z).$$

Since this surjection factors through $H^0(X, \mathcal{F}_3 \otimes f^*Q \otimes \mathcal{I}_x)$, we have that $\mathcal{F}_3 \otimes f^*Q$ separates tangent vectors at general points for all $Q \in \mathrm{Pic}^0(A)$.

Since $\mathcal{F}_3 = \mathcal{O}_X(4K_X) \otimes \tau(\|3\Delta\|)$ and $\tau(\|3\Delta\|)$ is an ideal, we can conclude that $|4K_X|$ induces a birational map. ■

CHAPTER 4

VOLUME OF ISOLATED SINGULARITIES

In this section, we will answer two questions about the non-log-canonical volume $\mathrm{Vol}_{\mathrm{BdFF}}$ defined by Boucksom, de Fernex and Favre.

Problem A Does there exist a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume?

Problem B Is it true that $\mathrm{Vol}_{\mathrm{BdFF}}(X,0) = 0$ implies the existence of an effective $\mathbb{Q}$-boundary Δ such that the pair (X,Δ) is log-canonical (the converse being easily shown)?

We will give a positive answer to Problem A and a counterexample of Problem B.

4.1 $\mathbb{Q}$-Gorenstein Case

Assume that X is a $\mathbb{Q}$-Gorenstein normal variety with isolated singularities. We pick $\Delta = 0$ and suppose that $f : Y \to X$ is the log canonical modification of X. Let $F = f^*K_X - K_Y - E_f$. We define $\mathrm{Vol}(X) = -(K_Y + E_f - f^*K_X)^n$. By the Negativity Lemma (see [17, Lemma 3.39]), $F \geqslant 0$. Since $K_Y + E_f$ is f-ample and F is f-exceptional, we have that

$$\mathrm{Vol}(X) = -(K_Y + E_f - f^*K_X)^n = F \cdot (K_Y + E_f)^{n-1} \geqslant 0.$$

Remark 27 *This definition can be extended to the case of a $\mathbb{Q}$-Gorenstein normal variety X which has isolated non-log-canonical locus.*

Theorem 28 *If X is a $\mathbb{Q}$-Gorenstein normal variety which has isolated singularities, then* $\mathrm{Vol}_{BdFF}(X,0) = \mathrm{Vol}(X)$.

Proof. We show that $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$ is a $\mathbb{Q}$-Cartier b-divisor and equals to $\overline{A_{Y/X}}$ where $f : Y \to X$ is the log canonical modification of $(X,0)$. Then the theorem follows immediately.

We only need to show that on a high enough model $f' : Y' \to X$ which factors through $f : Y \to X$ via $g : Y' \to Y$, we have that $D = \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})_{f'}$ equals to $g^*A_{Y/X}$.

First, we show that $g^*A_{Y/X} \leqslant D$. Since (Y, E_f) is log canonical, we have that

$$\begin{aligned} & A_{Y'/X} - g^*A_{Y/X} \\ = \ & K_{Y'} + E_{f'} - f'^*K_X - g^*(K_Y + E_f - f^*K_X) \\ = \ & K_{Y'} + (E_f)_{Y'} + E_g - g^*(K_Y + E_f) \\ \geqslant \ & 0, \end{aligned}$$

where $(E_f)_{Y'}$ is the strict transform of E_f on Y'. As $A_{Y/X}$ is f-ample, we have that $g^*A_{Y/X}$ is f'-nef. We can conclude that $\overline{A_{Y/X}}$ is a relatively nef $\mathbb{Q}$-Cartier b-divisor such that $\overline{A_{Y/X}} \leqslant A_{\mathcal{X}/X}$. By the definition of nef envelope, we have that $\overline{A_{Y/X}} \leqslant \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. In particular, $g^*A_{Y/X} \leqslant D$.

On the other hand, by the definition of nef envelope, we have that $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$ is relatively nef over X. We may apply Lemma 15. Thus, $D \leqslant -g^*(-\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})_f)$. By definition, $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})_f \leqslant A_{Y/X}$. Hence, $D \leqslant -g^*(-A_{Y/X}) = g^*A_{Y/X}$, since $A_{Y/X}$ is $\mathbb{Q}$-Cartier. Therefore, $D = g^*A_{Y/X}$. ■

We are able to answer Problem A.

Theorem 29 *There exists a positive lower bound, only depending on the dimension, for the volume of isolated Gorenstein singularities with positive volume.*

Proof. Suppose X is a Gorenstein normal variety with isolated singularities and $f : Y \to X$ is its log canonical modification. Let $F = f^*K_X - K_Y - E_f = \sum a_i \cdot E_i$, where E_i are f-exceptional divisors. Since X is Gorenstein, we have that a_i are positive integers by Lemma 8. If $\mathrm{Vol}(X) > 0$, then $F \neq 0$, hence $F \geqslant E_f$. We have that

$$\mathrm{Vol}(X) \geqslant E_f \cdot (K_Y + E_f)^{n-1} = ((K_Y + E_f)|_{E_f})^{n-1} = (K_{E_f} + \mathrm{Diff}_{E_f}(0))^{n-1}.$$

Since (Y, E_f) is log canonical, by [16, 16.6], the coefficients of $\mathrm{Diff}_{E_f}(0)$ lie in $\{0, 1\} \cup \{1 - \frac{1}{m} | m \geqslant 2\}$, which is a DCC set. By [12, Theorem 1.3], we have that $(K_{E_f} + \mathrm{Diff}_{E_f}(0))^{n-1}$ lies in a DCC set. The theorem follows. ■

4.2 Non-$\mathbb{Q}$-Gorenstein Case

Let X be a normal variety which has only isolated singularities. For any integer $m \geqslant 2$, fix a log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(-mK_X))$. By Theorem 14, we can find a boundary

Δ such that $K^{\Delta}_{Y/X} = K_{m,Y/X}$. Let $f_{lc} : Y_{lc} \to X$ be the log canonical modification of the pair (X, Δ). Then

$$Y_{lc} \cong \mathbf{Proj}_X \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} f_* \mathcal{O}_Y(m(K_Y + \Delta_Y + E_f)).$$

Assuming that Δ' is another m-compatible boundary for X with respect to f and $f'_{lc} : Y'_{lc} \to X$ is the corresponding log canonical modification, we have that $K^{\Delta}_{Y/X} = K^{\Delta'}_{Y/X}$. Hence, $\Delta_Y - \Delta'_Y = f^*(\Delta - \Delta')$. Now,

$$\begin{array}{rl} & f_*\mathcal{O}_Y(m(K_Y + \Delta'_Y)) \\ = & f_*\mathcal{O}_Y(m(K_Y + \Delta_Y - f^*(\Delta - \Delta'))) \\ = & f_*\mathcal{O}_Y(m(K_Y + \Delta_Y)) \otimes \mathcal{O}_X(m(\Delta' - \Delta)) \end{array}$$

for sufficiently divisible m, as $\Delta - \Delta'$ is $\mathbb{Q}$-Cartier. Thus, there is a natural X-isomorphism $\sigma : Y_{lc} \to Y'_{lc}$ such that $f_{lc} = f'_{lc} \circ \sigma$. Fix a common resolution of Y and Y_{lc}, $\tilde{f} : \tilde{Y} \to X$, as in the following diagram:

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ Y \arrow[rr, dashed] \arrow[dr, "f"'] & & Y_{lc} \arrow[dl, "f_{lc}"] \\ & X & \end{tikzcd}$$

Noticing that $\tilde{Y}$ is also a common resolution of Y and Y'_{lc}, we have that the morphism $s : \tilde{Y} \to Y$ is independent of the choice of Δ.

Theorem 30 *The $\mathbb{R}$-Weil b-divisor $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ is a $\mathbb{Q}$-Cartier b-divisor. If Δ is m-compatible for X with respect to $\tilde{f} : \tilde{Y} \to X$, then*

$$\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) = \overline{A^{\Delta}_{Y_{lc}/X}}.$$

Proof. We will mimic the proof of Theorem 28. We only need to show that on a high enough model $\rho : Z \to X$ which factors through $\tilde{f} : \tilde{Y} \to X$ by $\pi : Z \to \tilde{Y}$, we have that $D = \mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})_\rho$ equals to $(t \circ \pi)^* A^{\Delta}_{Y_{lc}/X}$.

First, we show that $(t \circ \pi)^* A^{\Delta}_{Y_{lc}/X} \leqslant D$. Since Δ is m-compatible for X with respect to $\tilde{f}$, we have that

$$K_{\tilde{Y}} + \Delta_{\tilde{Y}} - \tilde{f}^*(K_X + \Delta) = K_{\tilde{Y}} - \frac{1}{m}\tilde{f}^{\natural}(mK_X).$$

Hence,

$$-\frac{1}{m}\rho^{\natural}(mK_X) = \pi^*\left(-\frac{1}{m}\tilde{f}^{\natural}(mK_X)\right) = \pi^*\Delta_{\tilde{Y}} - \rho^*(K_X + \Delta),$$

by Lemma 12 and the fact that $\tilde{Y}$ is smooth. We calculate the difference

$$\begin{aligned}
& A_{m,Z/X} - (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X} \\
= \ & K_Z + E_\rho - \frac{1}{m}\rho^\natural(mK_X) - (t\circ\pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) + \rho^*(K_X+\Delta) \\
= \ & K_Z + E_\rho + \pi^*\Delta_{\tilde{Y}} - (t\circ\pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) \\
= \ & (K_Z + \Delta_Z + (E_{f_{lc}})_Z + E_{t\circ\pi} - (t\circ\pi)^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})) + (\pi^*\Delta_{\tilde{Y}} - \Delta_Z) \\
= \ & A^{\Delta_{Y_{lc}}+E_{f_{lc}}}_{Z/Y_{lc}} + (\pi^*\Delta_{\tilde{Y}} - \Delta_Z).
\end{aligned}$$

Since $(Y_{lc}, \Delta_{Y_{lc}} + E_{f_{lc}})$ is log canonical, we have that the first term $A^{\Delta_{Y_{lc}}+E_{f_{lc}}}_{Z/Y_{lc}}$ is effective and $(t\circ\pi)$-exceptional. As Δ_Z is the strict transform of the effective divisor $\Delta_{\tilde{Y}}$, the second term $\pi^*\Delta_{\tilde{Y}} - \Delta_Z$ is effective and π-exceptional, hence, $(t\circ\pi)$-exceptional. We conclude that $A_{m,Z/X} - (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$ is effective and $(t\circ\pi)$-exceptional. Since $A^{\Delta}_{Y_{lc}/X}$ is f-ample, we have that $(t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$ is ρ-nef. We can conclude that $\overline{A^{\Delta}_{Y_{lc}/X}}$ is a relatively nef $\mathbb{Q}$-Cartier b-divisor such that $\overline{A^{\Delta}_{Y_{lc}/X}} \leqslant A_{m,\mathcal{X}/X}$. By the definition of nef envelope, we have that $\overline{A^{\Delta}_{Y_{lc}/X}} \leqslant \operatorname{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$. In particular, $(t\circ\pi)^* A^{\Delta}_{Y_{lc}/X} \leqslant D$.

On the other hand, by the definition of nef envelope, we have that $\operatorname{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ is relatively nef over X. We may apply Lemma 15. Thus, $D \leqslant -(t\circ\pi)^*(-\operatorname{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})_{f_{lc}})$. By definition, $\operatorname{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})_{f_{lc}} \leqslant A_{m,Y_{lc}/X}$. Hence, $D \leqslant -(t\circ\pi)^*(-A_{m,Y_{lc}/X}) = (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$, since $A^{\Delta}_{Y_{lc}/X}$ is $\mathbb{Q}$-Cartier. Therefore, $D = (t\circ\pi)^* A^{\Delta}_{Y_{lc}/X}$. ∎

We can define the volume of isolated singularities of X as follow:

Definition 31 *The m-th limiting volume of singularity of X is*

$$\operatorname{Vol}_m(X) = -\operatorname{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})^n.$$

Corollary 32 *In the setting of Theorem 30, if Δ is m-compatible for X with respect to $\tilde{f}$, then*

$$\operatorname{Vol}_m(X) = -(A^{\Delta}_{Y_{lc}/X})^n = -A^{\Delta}_{Y_{lc}/X} \cdot (K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})^{n-1} \geqslant 0.$$

Proof. The first equation is straightforward by Theorem 30 and the definition of intersection number. The second equation is valid since $A^{\Delta}_{Y_{lc}/X}$ is f_{lc}-exceptional. By the Negativity Lemma, we have that $A^{\Delta}_{Y_{lc}/X} \leqslant 0$. Since $K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}$ is f_{lc}-ample, we have the inequality in the corollary. ∎

For an arbitrary boundary Δ on X, we have the following inequalities.

Proposition 33 *Suppose that Δ is a boundary on X, m is the index of $K_X + \Delta$ and $f : Y \to X$ is the log canonical modification of (X, Δ). Then*

1. $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \geqslant \overline{A^{\Delta}_{Y/X}}$,

2. $\mathrm{Vol}_m(X) \leqslant -(A^{\Delta}_{Y/X})^n$.

Proof. For any model $\pi : X_\pi \to X$, we have that

$$\pi^*(m(K_X + \Delta)) + \pi^\natural(-m\Delta) = \pi^\natural(mK_X).$$

Hence,

$$\begin{aligned} A_{m,X_\pi/X} &= K_{X_\pi} + E_\pi - \frac{1}{m}\pi^\natural(mK_X) \\ &= K_{X_\pi} + E_\pi - \frac{1}{m}\pi^\natural(-m\Delta) - \frac{1}{m}\pi^*(m(K_X + \Delta)) \\ &\geqslant K_{X_\pi} + E_\pi + \Delta_{X_\pi} - \pi^*(K_X + \Delta) = A^{\Delta}_{X_\pi/X}. \end{aligned}$$

Thus, as b-divisors, $A_{m,\mathcal{X}/X} \geqslant A^{\Delta}_{\mathcal{X}/X}$, hence,

$$\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \geqslant \mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}).$$

On the other hand, for any model $f' : Y' \to Y$ factoring through f via $g : Y' \to Y$, we have that $A^{\Delta}_{Y'/X} \geqslant g^* A^{\Delta}_{Y/X}$, since $(Y, \Delta_Y + E_f)$ is log canonical. As b-divisors, $A^{\Delta}_{\mathcal{X}/X} \geqslant \overline{A^{\Delta}_{Y/X}}$. As $K_Y + \Delta_Y + E_f$ is f-ample, we have that $\overline{A^{\Delta}_{Y/X}}$ is relatively nef over X. Thus, $\mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}) \geqslant \overline{A^{\Delta}_{Y/X}}$. We proved the first statement.

Since both $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ and $\overline{A^{\Delta}_{Y/X}}$ are relatively nef and exceptional over X, the second statement follows from the inequality between intersection numbers. ∎

Remark 34 *In the last proposition, one can show that* $\mathrm{Env}_{\mathcal{X}}(A^{\Delta}_{\mathcal{X}/X}) = \overline{A^{\Delta}_{Y/X}}$.

For any two positive integers m and l and any model $f : Y \to X$, since

$$\frac{1}{m} f^\natural(mK_X) \geqslant \frac{1}{lm} f^\natural(lmK_X) \geqslant f^* K_X,$$

we have that

$$A_{m,Y/X} \leqslant A_{lm,Y/X} \leqslant A_{Y/X},$$

hence,

$$A_{m,\mathcal{X}/X} \leqslant A_{lm,\mathcal{X}/X} \leqslant A_{\mathcal{X}/X}.$$

By the definition of nef envelope, we have that

$$\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \leqslant \mathrm{Env}_{\mathcal{X}}(A_{lm,\mathcal{X}/X}) \leqslant \mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X}).$$

Since they are both exceptional over X by Theorem 30, we have the following inequality of volumes:

$$\mathrm{Vol}_m(X) \geqslant \mathrm{Vol}_{lm}(X) \geqslant \mathrm{Vol}_{\mathrm{BdFF}}(X, 0).$$

Corollary 35 *The following statements are equivalent:*

1. *There exists a boundary* Δ *on* X *such that* (X, Δ) *is log canonical.*

2. $\mathrm{Vol}_m(X) = 0$ *for some (hence any multiple of) integer* $m \geqslant 1$.

Proof. (1) $\Rightarrow$ (2). Suppose $m(K_X + \Delta)$ is Cartier. By Proposition 33,

$$A_{m,\mathcal{X}/X} \geqslant A^{\Delta}_{\mathcal{X}/X} \geqslant 0,$$

since (X, Δ) is log canonical. As 0 is a relatively nef b-divisor over X, we have that $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \geqslant 0$. On the other hand, by Theorem 30 and the Negativity Lemma, $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) \leqslant 0$. Hence, $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X}) = 0$. We can conclude that $\mathrm{Vol}_m(X) = 0$.

(2) $\Rightarrow$ (1). Let Δ be an m-compatible boundary for X with respect to $\tilde{f}$ in the setting of Theorem 30. By Theorem 14, such a boundary always exists. Since $\mathrm{Vol}_m(X) = 0$, by Corollary 32, we have that

$$-A^{\Delta}_{Y_{lc}/X} \cdot (K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}})^{n-1} = 0.$$

Since $K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}$ is f_{lc}-ample, this is equivalent to $A^{\Delta}_{Y_{lc}/X} = 0$. Thus, we have that

$$f_{lc}^*(K_X + \Delta) = K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}.$$

For any model $\rho : Z \to X$ factoring through f_{lc} via $\pi : Z \to Y_{lc}$, we have that

$$\begin{aligned} A^{\Delta}_{Z/X} &= K_Z + \Delta_Z + E_\rho - \rho^*(K_X + \Delta) \\ &= K_Z + \Delta_Z + (E_{f_{lc}})_Z + E_\pi - \pi^*(K_{Y_{lc}} + \Delta_{Y_{lc}} + E_{f_{lc}}) \\ &\geqslant 0, \end{aligned}$$

since $(Y_{lc}, \Delta_{Y_{lc}} + E_{f_{lc}})$ is log canonical. Therefore, (X, Δ) is log canonical. ∎

Definition 36 *The augmented volume of singularities on* X *is*

$$\mathrm{Vol}^+(X) = \liminf_m \mathrm{Vol}_m(X) = \lim_{k\to\infty} \mathrm{Vol}_{k!}(X) \geqslant \mathrm{Vol}_{BdFF}(X, 0).$$

Remark 37 *While it is proved in the appendix of [3] that the intersection number is continuous, it is not clear that* $\mathrm{Env}_{\mathcal{X}}(A_{m,\mathcal{X}/X})$ *converge to* $\mathrm{Env}_{\mathcal{X}}(A_{\mathcal{X}/X})$. *It is interesting to have an example with* $\mathrm{Vol}^+(X) > \mathrm{Vol}_{BdFF}(X, 0)$.

4.2.1 Cone singularities

We will give a counterexample to Problem B in this section.

Let (V,H) be a nonsingular projective polarized variety of dimension $n-1$. The vertex 0 is the isolated singularity of the variety

$$X = \operatorname{Spec} \bigoplus_{m\geqslant 0} H^0(V, \mathcal{O}_V(mH)).$$

We assume that H is sufficiently ample so that X is normal. Blowing up 0 gives a resolution of singularities for X that we denote by Y. The induced map $f: Y \to X$ is isomorphic to the contraction of the zero section E of the total space of the vector bundle $\mathcal{O}_V(H)$. Let $\pi: Y \to V$ be the bundle map. We have that $E \cong V$. The co-normal bundle of E in Y is

$$\mathcal{O}_E(-E) \cong \mathcal{O}_V(H).$$

Let us slightly change our notation from previous sections. Let Γ be a boundary on X, Γ_Y be the strict transform of Γ on Y and $\Delta = \Gamma_Y|_E$. Since $A^{\Gamma}_{Y/X}$ is exceptional, we may assume that $A^{\Gamma}_{Y/X} = -aE$ for some rational number a. Restricting to E, we have that

$$K_V + \Delta \sim_{\mathbb{Q}} aH,$$

by the adjunction formula. On the other hand, assuming that Δ is an effective $\mathbb{Q}$-Cartier divisor on V such that $\Delta \sim_{\mathbb{Q}} -K_V + aH$, we may set $\Gamma = C_\Delta$ and get that $A^{C_\Delta}_{Y/X} = -aE$, where C_Δ is the cone over Δ in X.

Let C be an elliptic curve, U be a nonsplitting semistable vector bundle on C of rank 2 and degree 0 and $V = \mathbb{P}(U)$ be the ruled surface over C. The nef cone $\operatorname{Nef}(V)$ and pseudo-effective cone $\overline{\operatorname{NE}}(V)$ are the same. They are spanned by the section C_0 corresponding to the tautalogical bundle $\mathcal{O}_{\mathbb{P}(U)}(1)$ and a fiber F of the ruling (for details, see e.g. [18, Section 1.5.A]). Moreover, as in [24, Example 1.1], if C' is an effective curve on V such that $C' \equiv mC_0$ for some positive integer m, then $C' = mC_0$.

Theorem 38 *Let V be the ruled surface as above. Fix an ample divisor H on V. Let X be the affine cone over (V,H). Suppose H is sufficiently ample so that X is normal. Then* $\operatorname{Vol}^+(X) = 0$, *hence* $\operatorname{Vol}_{BdFF}(X,0) = 0$. *But there is no effective $\mathbb{Q}$-divisor Γ such that (X,Γ) is log canonical.*

Proof. Fix an ample divisor H on V. Since $K_V \sim -2C_0$, we have that $-K_V + aH$ is ample for any rational number $a > 0$. Let D be a smooth curve in $|n(-K_V + aH)|$ for some sufficiently large positive integer n, and set $\Delta = \frac{1}{n}D$. Then $(Y, \pi^*\Delta + E)$ is the log canonical modification of (X, C_Δ), since $(K_Y + \pi^*\Delta + E)|_E \sim_{\mathbb{Q}} aH$ is ample and (X, C_Δ) is not log canonical. Suppose m is the index of $K_X + C_\Delta$. By Proposition 33 (2),

$$\mathrm{Vol}_m(X) \leqslant -(A^{C_\Delta}_{Y/X})^3 = (aE)^3 = a^3 H^2.$$

As $a \to 0$, we conclude that $\mathrm{Vol}^+(X) = 0$, hence $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$.

If (X, Γ) is log canonical for some effective $\mathbb{Q}$-divisor Γ on X, then $A^{\Gamma}_{Y/X} = -aE \geqslant 0$, hence $a \leqslant 0$. On the other hand, let $\Delta = \Gamma_Y|_E \in \overline{\mathrm{NE}}(V)$. Then

$$\Delta \sim_{\mathbb{Q}} -K_V + aH \sim_{\mathbb{Q}} 2C_0 + aH.$$

Since $\Delta \geqslant 0$, we have $a \geqslant 0$ and hence $a = 0$. Thus $A^{\Gamma}_{Y/X} = 0$, and $(Y, \Gamma_Y + E)$ is log canonical. But $\Delta = \Gamma_Y|_E$ is an effective $\mathbb{Q}$-divisor linearly equivalent to $2C_0$, and hence $\Delta = 2C_0$, a contradiction. ■

In [6, Definition 7.1], a normal variety X is defined to be log canonical if for one (hence any sufficiently divisible) positive integer m, the m-th limiting log discrepancy b-divisor $A_{m,\mathcal{X}/X} \geqslant 0$. And in [ibid, Proposition 7.2], they proved that X is log canonical if and only if there is a boundary Δ such that the pair (X, Δ) is log canonical. It is natural to ask whether this definition is equivalent to the one requiring that the log discrepancy b-divisor $A_{\mathcal{X}/X} \geqslant 0$.

Corollary 39 *Let X be the affine cone over (V, H) as in Theorem 38. Then $A_{\mathcal{X}/X} \geqslant 0$, but X is not log canonical.*

Proof. The corollary follows immediately from the fact that $A_{\mathcal{X}/X} \geqslant 0$ is equivalent to $\mathrm{Vol}_{\mathrm{BdFF}}(X, 0) = 0$ (see [3, Proposition 4.19] or Corollary 35). ■

REFERENCES

[1] M. Blickle, M. Mustaţă, and K. E. Smith, *Discreteness and rationality of F-thresholds*, Michigan Math. J., 57 (2008), pp. 463–483.

[2] E. Bombieri, *Canonical models for surfaces of general type*, Inst. Hautes Études Sci. Publ. Math., 72 (1973), pp. 127–219.

[3] S. Boucksom, T. de Fernex, and C. Favre, *The volume of an isolated singularity*, Duke Math. J., 161 (2012), pp. 1455–1520.

[4] A. Broustet and A. Höring, *Singularities of varieties admitting an endomorphism.* Preprint, arXiv:1210.6254.

[5] J. A. Chen and C. Hacon, *Linear series of irregular varieties*, in Proceedings of the symposium on Algebraic Geometry in East Asia, World Scientific, 2002, pp. 143–153.

[6] T. de Fernex and C. Hacon, *Singularites on normal varieties*, Compos. Math., 145 (2009), pp. 393–414.

[7] Y. Fujimoto and N. Nakayama, *Compact complex surfaces admitting non-trivial surjective endomorphisms*, Tohoku Math. J., 57 (2005), pp. 395–426.

[8] M. Fulger, *Local volumes on normal algebraic varieties.* To appear in Annales de l'Institut Fourier.

[9] F. Ganter, *Properties of $-p \cdot p$ for Gorenstein surface singularities*, Math. Z., 223 (1996), pp. 411–419.

[10] C. Hacon, *Singularities of pluri-theta divisors in Char $p > 0$.* To appear in Alg. Geom. East Asia.

[11] C. Hacon and J. McKernan, *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math., 166 (2006), pp. 1–25.

[12] C. Hacon, J. McKernan, and C. Xu, *ACC for log canonical thresholds.* Preprint, arXiv:1208.4150.

[13] R. Hartshorne, *Algebraic Geometry*, no. 52 in Graduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1977.

[14] Z. Jiang, *On varieties of maximal Albanese dimension.* Preprint, arXiv:0909.4817.

[15] Z. Jiang, M. Lahoz, and S. Tirabassi, *On the Iitaka fibration of varieties of maximal Albanese dimension*, Int. Math. Res. Not., (2013), pp. 2984–3005.

[16] J. Kollár et al., *Flips and abundance for algebraic threefolds*, no. 211 in Astérisque, Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at University of Utah, Salt Lake City, Utah, August 1991.

[17] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, vol. 134 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1998.

[18] R. Lazarsfeld, *Positivity in algebraic geometry I*, vol. 48 of A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series], Springer-Verlag, Berlin, 2004.

[19] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J., 81 (1981), pp. 153–175.

[20] M. Mustaţă, *The non-nef locus in positive characteristic*, in A celebration of algebraic geometry, vol. 18 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2013, pp. 535–551.

[21] N. Nakayama, *On complex normal projective surfaces admitting non-isomorphic surjective edomorphisms*. Preprint.

[22] F. Odaka and C. Xu, *Log-canonical models of singular pairs and its applications*, Math. Res. Lett., 19 (2012), pp. 325–334.

[23] K. Schwede, *A canonical linear system associated to adjoint divisors in characteristic $p > 0$*. To appear in Journal für die reine und angewandte Mathematik.

[24] V. Shokurov, *Complements on surfaces*, J. Math. Sci. (New York), 102 (2000). Algebraic geometry 10.

[25] S. Takayama, *Pluricanonical systems on algebraic varieties of general type*, Invent. Math., 165 (2006), pp. 551–587.

[26] H. Tsuji, *Pluricanonical systems of projective varieties of general type. I*, Osaka J. Math., 43 (2006), pp. 967–995.

[27] ——, *Pluricanonical systems of projective varieties of general type. II*, Osaka J. Math., 44 (2007), pp. 723–764.

[28] S. Urbinati, *Discrepancies of non-$\mathbb{Q}$-Gorenstein varieties*, Michigan Math. J., 61 (2012), pp. 265–277. Issue 2.

[29] J. Wahl, *A characteristic number for links of surface singularities*, J. Amer. Math. Soc., 3 (1990), pp. 625–637.

[30] Y. Zhang, *On the volume of isolated singularities*. To appear in Compos. Math.

[31] ——, *Pluri-canonical maps of varieties of maximal Albanese dimension in positive characteristic*. Preprint, arXiv:1203.1360.